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**Reviewer:** Arimoto, Akio

**Reviewer number:** 49531

**Address:**
Department of Mathematics
Tokyo City University
Setagaya 158-8557, Tokyo
JAPAN
arimoto@s9.dion.ne.jp, arimoto@iname.com

**Author:** Valusescu, Ilie

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**Review text:**

Gladyshev’s works(1961) on a periodically correlated process are generalized to the operator model. These generalizations are natural, and we could survey the whole thing of prediction theory. A linear operatorial Wiener filter for prediction is obtained and prediction error is evaluated by coefficients of the maximal outer function which goes back to Arne Beurling’s idea. Let $E$ be a separable Hilbert space, $L(E)$ the C*-algebra of all linear bounded operators on $E$ and $H$ a right $L(E)$-module. In the prediction terminology, $E$ is called the parameter space, and $H$ the state space. A sequence $\{f_t\}_{t \in \mathbb{Z}}$ in $H$ is said to be a periodically $\Gamma$-correlated process of period $T$ if it satisfied that $\Gamma[f_{t+T}, f_{s+T}] = \Gamma[f_t, f_s]$ for $t, s \in \mathbb{Z}$. In particular if $T=1$, $\{f_t\}_{t \in \mathbb{Z}}$ is simply $\Gamma$-stationary (or weakly stationary). Under the operator model, $\Gamma$ should be considered as a map from $H \times H$ into $L(E)$ such that

(i) $\Gamma[h, h] \geq 0, \Gamma[h, h] = 0$ implies $h = 0$

(ii) $\Gamma[g, h] = \Gamma[h, g] \ast$

(iii) $\sum A_i h_i, \sum B_j h_j = \sum A_i^* \Gamma[h_i, g_j] B_j$. When we have the correlation function of a periodically $\Gamma$-correlated process $\{f_t\}_{t \in \mathbb{Z}}$ with $\Gamma : \mathbb{Z} \times \mathbb{Z} \rightarrow L(E), \Gamma(t, s) = \Gamma[f_t, f_s]$, we can define the corresponding covariance function $B(n, t) = \Gamma(n + t, n)$. It is easily seen that $B(n, t)$ is a periodic function in the first argument $n$ with the period $T$ and has the Fourier representation $B(n, t) = \sum_{k=0}^{T-1} B_k(t) \exp(2\pi i kn/t)$ with coefficients $B_k(t)$ of $L(E)$ valued. Now a generalization of Gladyshev’s theorem is given: //**Theorem 1.**

A function $B(n, t)$ is the covariance function of some periodically $\Gamma$-correlated process in $H$ with period $T$, if and only if the $T \times T$ matrix valued function
cr\text{e} \times \mathcal{H}, \ Gamma is the right \\
\mathcal{L}(E)-\text{module and} \ Gamma : \mathcal{L}(E, K) \times \mathcal{L}(E, K) \to \mathcal{L}(E) \text{ given by} \Gamma [V_1, V_2] = V_1^* V_2 \text{ is a} \\
correlation of the action of } \mathcal{L}(E) \text{ on } \mathcal{L}(E, K). \{E, \mathcal{L}(E, K), Gamma\} \text{ is called an operator} \\
model. An abstract correlated action } \{E, \mathcal{H}, Gamma\} \text{ is imbedded into the operator} \\
model \{E, \mathcal{L}(E, K), Gamma\} \text{ by imbedding } h \to V_h, \text{ that is, considering for each } h \in \mathcal{H} \\
the operator } V_h \in \mathcal{L}(E, K) \text{ is given by } V_h(a) = \gamma(a, h). \text{ In the same way we have a} \\
unique ( up to a unitary equivalence) imbedding \mathcal{H}^T \to \mathcal{L}(E, K)^T \text{ by} \\
defining the operator } W_X = (V_{x_1}, V_{x_2}, \ldots, V_{x_T}) \text{ for } X = (x_0, x_1, \ldots, x_{T-1}) \in \mathcal{H}^T. \\
So, the correlated action } \{E, \mathcal{L}(E, K), Gamma\} \text{ can be extended to an appropriate operator} \\
model \{E, \mathcal{L}(E, K)^T, Gamma\}, \text{ where } W_i = (V_{i_1}^*, V_{i_2}^*, \ldots, V_{i_T}^*) \in \mathcal{L}(E, K)^T, \\
Gamma [W_1, W_2] = \left( Gamma \left[ V_{i_1}^* V_{j_1}^*, V_{i_2}^* V_{j_2}^* \right] \right)^T. \text{ A matrix with} \\
operator elements. Starting with the correlation action } \{E, \mathcal{H}, Gamma\} \text{ of } \mathcal{L}(E) \text{ on} \\
\mathcal{H}, \text{ we obtain the correlation actions of } L(E)^{T \times T} \text{ on } \mathcal{H}^T. \text{ An element } Z = \\
(W_1, \ldots, W_T) \text{ is a vector with } W_k \in \mathcal{L}(E, K)^T \text{ and } W_k = (V_{i_1}^*, V_{i_2}^*, \ldots, V_{i_T}^*) \in \mathcal{L}(E, K)^T. \text{ Letting E the operator of multiplying by } e^{-2\pi i T/k} \text{ we can} \\
attach T sequence } X_n^k = (E^{k n} f_n, E^{k(n+1)} f_{n+1}, \ldots, E^{k(n+T-1)} f_{n+T-1}) \text{, } k \in \\
\{0, 1, \ldots, T-1\}. \text{ Using the imbedding } h \to V_k \text{ and} \\
its extended imbedding } \mathcal{H}^T \to \mathcal{L}(E, K)^T: X \to W_X, \text{ we obtain T sequences } Z_n^k = W_X^k \in \mathcal{L}(E, K)^T, \text{ } k \in \\
\{0, 1, \ldots, T-1\}, \text{ and a sequence } Z_n = \frac{1}{\sqrt{T}} (Z_n^0, Z_n^1, \ldots, Z_n^{T-1}) \in \mathcal{L}(E, K)^T. \\
In this way we have another operator version of the Gladyschev's theorem: } \text{//Theorem2. A process } \{f_t\}_{t \in \mathbb{Z}} \text{ is a periodically } Gamma \text{-correlated with period } T \\
if and only if } Z_n = \frac{1}{\sqrt{T}} (Z_n^0, Z_n^1, \ldots, Z_n^{T-1}) \text{ is a stationary } Gamma \text{-correlated} \\
process. } \text{// Theorem3} \text{ in the article deals with the past and present structure of } \\
Z_n. \text{ The predictable part of } Z_{n+1} \text{ can be obtained by } \\
\hat{Z}_{n+1} = \sum_{k=0}^{\infty} A_k Z_{n-k}, \text{ where the Wiener filter } A_k = \left( A_{ij}^{(k)} \right)^{T-1}_{i,j=0} \text{ for prediction is given in terms of} \\
the coefficients of its maximal outer function in the similar way as the standard one } : \text{//Theorem4. Let } \{f_t\}_{t \in \mathbb{Z}} \text{ is a periodically } Gamma \text{-correlated process. Then the pre-}
dictable part \( \hat{f}_{n+1} = \sum_{k=0}^{\infty} C_k f_{n-k} \), where \( C_k = \sum_{j=0}^{T-1} \frac{1}{\sqrt{T}} A^j_k E_j(n-k) \) with \( A^j_k \) being the 0 line of \( A_k = \left( A^j_{ik} \right)_{i,j=0}^{T-1} \). // **Theorem 5.** The prediction error of a periodically \( \Gamma \)-correlated process \( \{f_t\}_{t \in \mathbb{Z}} \) has the form \( \Gamma \left[ f_{n+1} - \hat{f}_{n+1}, f_{n+1} - \hat{f}_{n+1} \right] = \sum_{k=0}^{T-1} D_k E_k(n+1) \), where the operator coefficients \( D_k \in \mathcal{L}(\mathcal{E}) \) are the entries of the zero line of the prediction error matrix of the attached stationary process \( \{Z_n\} \), namely \( D_k = \sum_{s=0}^{T-1} \Theta^*_s \Theta_{sk} \), where \( \Theta_{ij} = \Theta_{ij}(0) \) from the maximal function \( \Theta(\lambda) \) of the spectral measure of the process \( \{Z_n\} \). // To show the last theorem, we use the following facts. For the stationary process \( \{Z_n\} \), there exists an operator valued semispectral measure \( F \) on the torus \( T \) such that its correlation \( \{Z_n\} \) has the integral form, \( \Gamma(n) = \int_0^{2\pi} e^{-in\lambda} dF(\lambda) \). To each semispectral measure \( F \), it is attached a maximal \( L^2 \) bounded outer function \( \Theta(\lambda) \) such that \( F_\theta \leq F \). If for \( c > 0 \), Harnack type inequality \( c d\lambda \leq F \leq c^{-1} d\lambda \) hold, then the maximal function of the process has a bounded inverse \( \Omega(\lambda) \), and we can make the Wiener filter \( A_j = \sum_{p=0}^{j} \Theta_{p+1} \Omega_{j-p} \), \( \Theta(\lambda) = \Theta_0 + \Theta_1 \lambda + \cdots \), \( \Omega(\lambda) = \Omega_0 + \Omega_1 \lambda + \cdots \).  

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