s):

Review text:

Let $\mathbb{C}$ be the complex plane and $z \in \mathbb{C}$. In the previous paper, the authors have proved the truncated moment problem with the monomial $z^m \bar{z}^n$ with a pair of nonnegative integers $(m, n)$ ranging over an symmetric index set $T$, i.e. $(n, m) \in T$ for all $(m, n) \in T$, which says that if $\Delta = \{(n, n) : n \geq 0\} \subset T$, and $Z \subset \mathbb{C}$ be a determining set for polynomials $\mathbb{C}[z, \bar{z}]$, then the conditions (a) and (b) are equivalent: (a) there exists a positive Borel measure $\mu$ on $\mathbb{C}$ such that for any system of complex numbers $\{c_{m,n}\}(m,n)\subset T$ satisfies

$$
\int_{\mathbb{C}} z^m \bar{z}^n \mu(z) \, dz = c_{m,n}, \quad (m,n) \in T,
$$

and (b) $\sum_{(m,n)\subset T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}(m,n)\subset \mathbb{C}$ that satisfies $\sum_{(m,n)\subset T} p_{m,n} z^m \bar{z}^n \geq 0$ for all $z \in Z$. We put a slightly different condition (b') $\sum_{(m,n)\subset T} p_{m,n} c_{m,n} \geq 0$ for every finite system $\{p_{m,n}\}(m,n)\subset \mathbb{C}$ such that $\sum_{(m,n)\subset T} p_{m,n} z^m \bar{z}^n \geq 0$ for all $z \in Z$. Regarding (b) and (b') the authors discuss the following two questions: (i) Given a symmetric set $T$ obeying $\Delta \subset T$ and a nonzero system $\{c_{m,n}\}(m,n)\subset T$, is (b) equivalent to (b')? (ii) Given a symmetric $T$ obeying $\Delta \subset T$ and a closed proper subset $Z \subset \mathbb{C}$, is (b) equivalent to (b')? The answer to question (i) is shown to be negative regardless of the choice of $T$ and $\{c_{m,n}\}(m,n)\subset T$. On the other hand the answer to question (ii) depends essentially on the interplay between the sets $T$ and $Z$. In fact, denoting $\Gamma(\alpha) \equiv \{\rho e^{it} : t \in [0, \alpha], \rho \geq 0\}$, they show that the answer is yes for the following four cases: (1) $T$ is arbitrary, $\{|z|^2 : z \in Z\}$ is a proper subset of $[0, \infty]$, (2) $T \supset \Delta \cup \{(k,l),(l,k)\}, \; l-k \geq 1$ and there exists $\lambda$ such that $\{z \in \mathbb{C} : z^{l-k} = \lambda\} \subset \mathbb{C}\setminus Z$, (3) $T \supset \Delta \cup \{(k,l),(l,k)\}, \; l-k \geq 1$ and $Z \subset \Gamma(\alpha), \; \alpha \in \left[0, \frac{2\pi}{l-k}\right]$, and (4) $T \supset \Delta \cup \{(k,k+1),(k+1,k)\}, \; k \geq 0$. On the contrary they show that the answer is
no for the following cases: (1) $T = \Delta \cup \{(k, l), (l, k)\}, \ l-k \geq 2, Z \supset \Gamma \left( \frac{2\pi}{l-k} \right)$,
and (2) $T = \Delta, \ \{ |z|^2 : z \in Z \} = [0, \infty)$. Finally the case of $\Delta \not\subset T$ is also discussed.