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Short title: Elliptical range theorems for generalized numerical ranges of quadratic operators.

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Review text:

It is shown that for a given quadratic operator, the rank-k numerical range, the essential numerical range and q-numerical range are elliptical disks; the c-numerical range is a sum of elliptical disks, and the Davis-Wielandt shell is an ellipsoid with or without interior. Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$. The numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined by $W(A) = \{\langle Ax, x \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1 \}$. A non-scalar operator $A \in \mathcal{B}(\mathcal{H})$ is called a quadratic operator if there are $a, b \in \mathbb{C}$ such that $(A - aI)(A - bI) = 0$. For a positive integer $k \leq \dim \mathcal{H}$, the rank-k numerical range of $A \in \mathcal{B}(\mathcal{H})$ is defined by $\Lambda_k(A) = \{\lambda \in \mathbb{C} : \exists P \text{ orthogonal}, PAP = \lambda P \}$, for some rank-k orthogonal projection $P$. It is proved that if $A$ is a quadratic operator, $\Lambda_k(A)$ can be an empty set, a singleton, a line segment or an elliptical disk with all or none of its boundary. For $k = \infty$, we have $\Lambda_\infty(A) = W_c(A)$, where $W_c(A) = \cap \{\text{cl} (W(A + F)) : F \in \mathcal{B}(\mathcal{H}) \text{ has a finite rank} \}$ being called an essential numerical range. The Davis-Wielandt shell of $A$ is defined by $DW(A) = \{(\langle Ax, x \rangle, \langle Ax, Ax \rangle) : x \in \mathcal{H}, \langle x, x \rangle = 1 \}$. It is shown that a necessary sufficient condition is provided for $DW(A)$ to be the prescribed closed ellipsoid. For $q \in [0, 1]$, the q-numerical range of $A$ is defined by $W_q(A) = \{\langle Ax, x \rangle, \langle Ax, y \rangle : x, y \in \mathcal{H}, \langle x, x \rangle = \langle y, y \rangle = 1, \langle x, y \rangle = q \}$. It is shown that for a quadratic operator $A$, $W_q(A) = W_q(A_0)$ or $W_q(A) = \text{int} W_q(A_0)$, where $A_0$ is a $2 \times 2$ matrix \[
\begin{bmatrix}
a & ||P|| \\
0 & b
\end{bmatrix} \]
and $W(A_0)$ is well known to be an elliptical disk with $a, b$ as foci. For $c = (c_1, c_2, \cdots, c_k)$ with $c_1 \geq c_2 \geq \cdots \geq c_k$ and $k \leq \dim \mathcal{H}$, the c-numerical range of $A$ is defined by $W_c(A) = \left\{\sum_{j=1}^{k} c_j \langle Ax_j, x_j \rangle : \{x_1, x_2, \cdots, x_k \} \subset \mathcal{H} \text{ is an orthogonal set} \right\}$. 

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For a quadratic operator $A$, the form of $W_c(A)$ is depicted in detail.