Chapter 4 Past and Future pp.82-90

1. Kolmogorov-Wiener Prediction Problem

Fix a stationary Gaussian process $X$ with spectral function $\Delta$. The classical prediction problem of Kolmogorov and Wiener is to compute the distribution of $X(T)$ at a future time $T > 0$, conditional on the past $X(t): t \leq 0$. It is a common idea that the trigonometric isomorphism $X(t) \mapsto e^{i\omega t}$ is applied, which comes from the relation

$$E(X(t_1)X(t_2)) = \int_{-\infty}^{\infty} e^{i\omega t_1} e^{i\omega t_2} d\Delta(x).$$

The last relation means that the correlation among $X(t_1)$ and $X(t_2)$ is given by calculating the inner product of $e^{i\omega t_1}$ and $e^{i\omega t_2}$ in the space $H \equiv L^2(\mathbb{R}, d\Delta)$. In this language the problem of prediction is to project $e^{iTz}$ for fixed $T > 0$ onto the span of negative frequencies: $H_{-\infty, 0} = s\mathcal{P}\{e^{i\omega t} : t \leq 0\}L^2(\mathbb{R}, d\Delta)$, as this corresponds to the past of $X$. The manifolds $H_{a:b} = s\mathcal{P}\{e^{i\omega t} : a < t < b\}L^2(\mathbb{R}, d\Delta)$ are introduced. Manifolds of those types have properties such that

(1.1) $H_{a:b} = s\mathcal{P}\{e^{i\omega t} : a \leq t \leq b\}L^2(\mathbb{R}, d\Delta)$.

(1.2) $H_{-\infty, \infty} = L^2(\mathbb{R}, d\Delta)$.

(1.3) $H_{-\infty, b} = e^{i(b-a)\omega}H_{-\infty, a}$.

(1.4) $e^{i\omega t} \in H_{-\infty, a}$ either for every $b$ or else for no $b > a$; in the first case $H_{-\infty, a} = L^2(\mathbb{R}, d\Delta)$ for every $a > -\infty$. 


(1.5) \( H_{-\infty,a} = \bigcap_{b>a} H_{-\infty,b} \).

**Warning:** Generally \( H_{ab} = \bigcap_{n=1}^{\infty} H_{a-1/n,b+1/n} \).

2. Szegö’s Alternative

The problem whether the prediction is perfect or not, is interpreted in trigonometrical language as to decide if \( H_{-\infty,0} \) contains \( e^{ixT}, \ T > 0 \) or not. By (1.4), that takes place either for every \( T > 0 \) or for no \( T > 0 \), according as \( H_{(-\infty,0)} = L^2(\mathbb{R},d\Delta) \) or not. To prepare for the statement, split \( H = L^2(\mathbb{R},d\Delta) \) into the perpendicular sum of

\[ H' = L^2\left(\mathbb{R},\Delta'(x)\,dx\right) \quad \text{and} \quad H^* = L^2\left(\mathbb{R},d\Delta^*(x)\right), \]

hence \( H = H' \oplus H^* \). Here we introduced symbols \( \Delta^* \) the singular part of \( \Delta \) and \( \Delta' \) is the density of the nonsingular part of \( \Delta \). \( H^* = L^2\left(\mathbb{R},d\Delta^*(x)\right) \) is identified with the inhabitants of \( H = L^2(\mathbb{R},d\Delta) \) that vanish off the singular set of \( \Delta \). Define also the manifold \( H_{-\infty} = \bigcap_{t<0} H_{-\infty,t} \) which corresponds to the infinite past of the stationary Gaussian process \( X \). The decision as to perfect or imperfect prediction can now made with the help of:

**Theorem 2.1** Szegö’s Alternative

*Either*

\[ \int_{-\infty}^{\infty} \frac{\log \Delta'(x)}{1+x^2} \, dx > -\infty \quad \text{and} \quad H = H_{-\infty,0} = H_{-\infty} = H^* \]

*or*

\[ \int_{-\infty}^{\infty} \frac{\log \Delta'(x)}{1+x^2} \, dx = -\infty \quad \text{and} \quad H = H_{-\infty,0} = H_{-\infty}. \]

**Historical Amplification:** The original statement of Szegö[5] is that if \( \Delta(\theta) \) is an
increasing function on the circle \( 0 \leq \theta < 2\pi \), then
\[
\inf \int_0^{2\pi} \left| 1 - \sum_{n < 0} c_n e^{i n \theta} \right|^2 \Delta \left( d\theta \right) = \exp \left\{ \int_0^{2\pi} \Delta \left( \theta \right) d\theta \right\}.
\]

A more refined alternative incorporating the identification of \( H_{\infty} = \bigcap_{t < 0} H_{-\infty,t} \) and
\[
H^* = L^2 \left( \mathbb{R}, d\Delta^\prime \left( x \right) \right) \text{ if } \int_0^{2\pi} \Delta \left( \theta \right) d\theta > -\infty \text{ is due to Kolmogorov [4]. The present alternative was proved by Krein[3] and Wiener[6] independently; see Akhiezer [1,pp.256-266] for a proof in the style of approximation theory.}

The proof is divided into two lemmas. \( P^a \) denotes the projection onto \( H_{-\infty,a} \).

**Lemma 2.2**

If \( H = H_{-\infty,0} \), then \( H_{-\infty,0} = H_{-\infty} = H^* \) and \( \int_{-\infty}^\infty \frac{\lg \Delta' \left( x \right)}{1 + x^2} dx > -\infty \).

If \( H = H_{-\infty,0} \), then you can find \( a < b \) so as to make \( f_{ab} = \left( 1 - P^a \right) e^{iab} = 0 \). Now \( f_{ab} \) is perpendicular to \( e^{i\sigma \ell} \) for \( \ell \leq a \), so
\[
\int_{-\infty}^\infty e^{-ixa} f_{ab} \left( x \right) e^{-ib\sigma} d\Delta \left( x \right) = 0
\]
for \( \ell \leq 0 \). Therefore, by the theorem of F. Riesz and M. Riesz, \( e^{-ixa} f_{ab} \) vanishes on the singular set of \( \Delta \), so \( e^{-ixa} f_{ab} \left( x \right) d\Delta \left( x \right) = e^{-ixa} f_{ab} \left( x \right) \Delta' \left( x \right) dx \), and \( e^{-ixa} f_{ab} \left( x \right) \Delta' \left( x \right) \) is of class \( H^{1+} \). In particular, \( \int_{-\infty}^\infty \frac{\lg \left| f_{ab} \left( x \right) \Delta' \left( x \right) \right|}{1 + x^2} dx > -\infty \), from which you deduce that
\[
\int_{-\infty}^\infty \frac{\lg \left| f_{ab} \left( x \right) \Delta' \left( x \right) \right|^2}{1 + x^2} dx - \int_{-\infty}^\infty \frac{\lg \left| \int_{-\infty}^\infty f_{ab} \left( x \right) \Delta' \left( x \right) dx \right|^2}{1 + x^2} dx
\]
\[
\geq 2 \int_{-\infty}^\infty \frac{\lg \left| f_{ab} \left( x \right) \Delta' \left( x \right) \right|}{1 + x^2} dx - \left\| f_{ab} \left( x \right) \right\|^2 \right\| > -\infty,
\]
also. To finish the present half of the proof, you have only to check that \( H_{-\infty} = H^* \). But
if \( f \) belongs to \( H_{-\infty} = H^* \), then so does \( e^{iat}f \) for every real \( t \). Moreover, for any \( a < b \), \( f_{ab} = (1 - P^a) e^{iab} \) is perpendicular to \( H_{-\infty,a} \supset H_{-\infty} \), and since

\[
f_{ab}(x)d\Delta(x) = f_{ab}(x)\Delta'(x)dx,
\]
we have

\[
\int_{-\infty}^{\infty} e^{iat} f_{ab}(x) d\Delta(x) = \int_{-\infty}^{\infty} e^{iat} f_{ab}(x) \Delta'(x) dx = 0
\]

for every \( t \). This means that \( f_{ab}(x) \Delta'(x) \) vanishes almost everywhere relative to \( dx \).

Therefore \( f \) does too because

\[
\int_{-\infty}^{\infty} \frac{\log|f_{ab}(x)\Delta'(x)|}{1 + x^2} dx > -\infty \quad \text{prevents} \quad f_{ab}(x) \Delta'(x)
\]

from vanishing almost everywhere. This proves \( H_{-\infty} \subset H^* \). To prove the opposite inclusion, choose \( f \in H^* \). Then, as \( f = 0 \) almost everywhere relative to \( dx \) and

\[
f_{ab}(x)d\Delta(x) = f_{ab}(x)\Delta'(x)dx,
\]

\[
\int_{-\infty}^{\infty} (1 - P^a) f(x) e^{-ixb} d\Delta(x) = \int_{-\infty}^{\infty} f(x) \left[(1 - P^a) e^{ixb}\right] d\Delta(x)
\]

\[
= \int_{-\infty}^{\infty} f(x) f_{ab}(x) \Delta'(x) dx = 0
\]

for every \( b \geq a \), which means that \( (1 - P^a) f \) is perpendicular to both \( H_{-\infty,a} \) and \( H_{-\infty} \) and, as such, must vanish. In other words, \( P^a f = f \in H_{-\infty,a} \) for every \( a \). This completes the present half of the proof.\( \blacksquare \)

**Lemma 2.3**

If \( H = H_{-\infty,0} \), then

\[
\int_{-\infty}^{\infty} \frac{\log\Delta'(x)}{1 + x^2} dx = -\infty \quad \text{and} \quad H_{-\infty,0} = H_{-\infty}
\]

Let \( H = H_{(-\infty,0)} \). We have \( H = \bigcap_{T < 0} e^{ixT} H_{-\infty,T} = H_{-\infty} \).
automatically, and it remains only to check that \( \int_{-\infty}^{\infty} \frac{\lg \Delta'(x)}{1 + x^2} dx > -\infty \) is not possible.

The convergence of the integral allows you to choose a function \( h \in H^2 \) so as to make \( |h|^2 = \Delta' \), and if you now declare \( \frac{h}{h} \) to be 1 on the singular set of \( \Delta \), it will follow from \( H = H(-\infty, 0) \) that if \( f \) denotes the general sum of the form \( c_1 e^{i t_1} + c_2 e^{i t_2} + \cdots + c_n e^{i t_n} \) with \( t_1, t_2, \ldots, t_n \leq 0 \), then

\[
\inf \int_{-\infty}^{\infty} \left| f(x)h(x) - h(x) \right|^2 dx \leq \inf \int_{-\infty}^{\infty} \left| f(x) - \frac{h(x)}{h(x)} \right|^2 d\Delta(x) = 0
\]

The upshot will be that \( h \) belongs to \( H^2 \cap H^2 = 0 \), contradictory

\[
\int_{-\infty}^{\infty} \frac{\lg \Delta'(x)}{1 + x^2} dx > -\infty
\]

The proof is finished.

3. Doing the Prediction

To solve the prediction problem we have to project \( e^{i\pi T} \) for fixed \( T > 0 \) onto \( H(-\infty, 0) \).

The simplest case is when \( \Delta \) is nonsingular and \( \int_{-\infty}^{\infty} \frac{\lg \Delta'(x)}{1 + x^2} dx > -\infty \). Then \( \Delta' \) can be expressed as \( |h|^2 \) with a function \( h \) of class \( H^2 \). You can even make \( h \) outer if you want and impose the reality condition \( \overline{h(x)} = h(-x) \) so as to make the Fourier inverse transform \( h^\vee \) real. Now if \( f \) denotes the general sum of the form \( c_1 e^{i t_1} + c_2 e^{i t_2} + \cdots + c_n e^{i t_n} \) with \( t_1, t_2, \ldots, t_n \leq 0 \), then the projection \( P e^{i\pi T} \) of \( e^{i\pi T} \) onto \( H(-\infty, 0) \) will satisfy

\[
||e^{i\pi T} - P e^{i\pi T}|| = \inf \int_{-\infty}^{\infty} \left| e^{i\pi T} - f(x) \right|^2 d\Delta(x)
\]

\[
= \inf \int_{-\infty}^{\infty} \left| e^{i\pi T} \overline{h(x)} - f(x) \overline{h(x)} \right|^2 dx
\]
and if you pick $h$ outer so as to make the span $e^{ixt} \overline{h}(x)$, $t \leq 0$ in $L^2(\mathbb{R}, dx)$ coincide with $H^2$. The second infimum will be achieved by the projection of $e^{ixT} \overline{h}(x)$ onto $H^2$

$$
\int_{-\infty}^{0} e^{ixt} \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixT} \overline{h}(x) e^{-ixt} dx \right] dt = \int_{-\infty}^{0} e^{ixt} \left[ h^\vee(T-t) \right] dt = e^{ixT} \int_{T}^{\infty} e^{-ixt} \left[ h^\vee(t) \right] dt.
$$

The conjugation is eliminated by imposing the reality condition on $h$, and the upshot is that the projection of $e^{ixT}$ onto $H_{(-\infty)}$ is $P e^{ixT} = e^{ixT} \overline{\frac{h}{h}(x)} \int_{0}^{\infty} e^{-ixt} h^\vee(t) dt$.

The norm of the coprojection (=mean square error by the prediction) is computed by the Plancherel identity:

$$
\left\| (1 - P) e^{ixT} \right\|_\Delta^2 = \int_{-\infty}^{\infty} e^{ixT} \int_{0}^{T} e^{-ixt} h^\vee(t) dt \right| _{dx}^2 = 2\pi \int_{0}^{T} |h^\vee(t)|^2 dt.
$$

Actually, we must see this more detail. Now suppose that $\Delta$ has a singular part but keep $\int_{-\infty}^{\infty} \log \Delta'(x) dx > -\infty$. Then $H_{-\infty} = H^\star$ is nontrivial, and $H_{-\infty}$ is perpendicular sum of $H_{-\infty}$ and the annihilator of the latter in $H_{-\infty}$:

$$
H_{-\infty} / H_{-\infty} = \text{span of } e^{ixt} : t \leq 0 \text{ in } H^\prime = L^2(\mathbb{R}, \Delta'(x) dx).$$

Now it is clear that if $P^\star$ is the projection onto $H^\ast = H_{-\infty}$ and $P$ is the projection onto $H_{(-\infty)}$, then

$$
P^\ast (1 - P) = 0,$$

so that

$$
\left\| (1 - P) e^{ixT} \right\|_\Delta^2 = \left\| P^\ast (1 - P) e^{ixT} + (1 - P^\ast)(1 - P) e^{ixT} \right\|_\Delta^2
$$

$$
= \left\| (1 - P) e^{ixT} \right\|_{\Delta^\prime} \int_{-\infty}^{\infty} \left\| (1 - P) e^{ixT} \right\|_\Delta^2 dx = 2\pi \int_{0}^{T} |h^\vee(t)|^2 dt
$$

just as if $\Delta^\prime = 0$. 

6
References

[5] Szego, G. Beiträge zur Theorie der Toeplitzen Formen, Math. Soc. 6 (1920) 167-202