We have to solve the problem such that

$$\dot{u}^2 + pu^2 + qu^4 + \varepsilon u^5 = 0,$$

where $u = u(z)$ and $\dot{u} = \dfrac{du(z)}{dz}$.

In the very old paper in 1856*, Briot and Bouquet have already solved such problems and have done much more. They showed, for example,

(1) $\dot{u}^3 + 3(u-2)^2\dot{u}^2 + \dfrac{243}{16}(u-1)^2(u-2)^4 - 4(u-2)^6 = 0$

has a solution

$$u = \frac{z + \frac{9z^2}{2} - \frac{9z^3}{4}}{1 + \frac{z}{2} + \frac{9z^2}{4} - \frac{9}{8}z^3} \quad \text{(a rational function).}$$

(2) $\dot{u}^3 - \dot{u}^2 - \dfrac{4}{27}\left(1 - 2u^2 + 2u^3\right)^2 + \dfrac{4}{27} = 0$

has a solution

$$u = \frac{\frac{\omega}{\pi}\tan\frac{\pi z}{\omega}\left(1 + \tan^2\frac{\pi z}{\omega}\right)}{1 + \frac{\omega}{\pi}\tan^2\frac{\pi z}{\omega}}, \quad \omega = \frac{3\sqrt{3}}{2}\pi i \text{ is a period,}$$

(a simply periodic function)

(3) $\dot{u}^3 - 3\dot{u}^2 - 2\left(u^2 - 1\right)^2 + 4 = 0$

has a solution

$$u = A\lambda^2 + B\lambda + C\lambda\dot{\lambda}$$

where $\lambda = \lambda(z)$ is an elliptic function and $\dot{\lambda} = \dfrac{d\lambda(z)}{dz}$ is also known to be elliptic function.

Their theorem is: Let $F\left(x,y\right)$ be a polynomial in x and y.

$F\left(u,\dot{u}\right)=0$ has only three type of the solutions,

(1) rational functions
(2)simply periodic functions
(3) doubly periodic function (constructed by λ and $\dot{\lambda}$)

They will solve our problem easily if they are alive now.

*)L'Integration des Equations Differentielles au Moyen des Fonctions Elliptiques
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