Elliptic Functions and Poncelet’s Theorem for confocal ellipses
November 22, 2011
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1. Introduction
We take up an example that elliptic functions play a crucial role in the Poncelet’s closure theorem. This example can be found in [2] (Example 7,p.141) and [3]. We will see that everything has already been done with the addition theorems in Cayley[1].

2. Setting the problem
Fix \( 0 < b^2 < a^2 \) and consider the family of confocal ellipses \( \frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1 \) \( (0 < c^2 < b^2) \) with common forci \( (\pm \sqrt{a^2-b^2},0) \). The tangent to the ellipse through the point \( (\xi,\eta) \) on it is

\[ \frac{\xi}{a^2} x + \frac{\eta}{b^2} y = 1. \]

We can describe all points on the plane in terms of the polar coordinate \( (x,y) = (r \text{cn}(u,k), r \text{sn}(u,k)) \), where \( r \geq 0 \), \( 0 \leq k < 1 \), \( 0 \leq u \leq \int_0^{2\pi} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = 4K \).

For \( k = 0 \), we only have usual polar coordinates \( (x,y) = (r \cos(u), r \sin(u)) \). From now on we omit \( k \) writing symbols in a simple way. Any point \( (\xi,\eta) \) on the ellipse could be denoted by \( Q = (\xi,\eta) = (a \text{cn}u, b \text{sn}u) \) and the tangent (1) becomes

\[ \frac{\text{cn}u}{a} x + \frac{\text{sn}u}{b} y = 1. \]

There are two intersection points of the tangent line (2) with \( x^2 + y^2 = R^2 \), denoted by \( P_1 = P(u) = (R \text{cn}u_1, R \text{sn}u_1) \) and \( P_2 = P(u) = (R \text{cn}u_2, R \text{sn}u_2) \). Since \( P_1, P_2 \) lie on the tangent line (2), it holds that
\[
\begin{align*}
(3) \quad & \frac{cn u}{a} cn u_1 + \frac{sn u}{b} sn u_1 = \frac{1}{R}, \\
\text{and} \quad & \frac{cn u}{a} cn u_2 + \frac{sn u}{b} sn u_2 = \frac{1}{R}.
\end{align*}
\]

From these equations we have, after a simple calculation,
\[
\begin{align*}
(5) \quad & \frac{a}{R} = \frac{cn u (sn u, cn u_2 - sn u, cn u_1)}{sn u_1 - sn u_2}, \\
(6) \quad & \frac{b}{R} = \frac{sn u (sn u, cn u_2 - sn u, cn u_1)}{cn u_2 - cn u_1}.
\end{align*}
\]

It seems that \(a\) and \(b\) must depend on \(u\) of the coordinate \(Q = Q(u) = (\xi, \eta) = (a cn u, b sn u)\). However if we set \(u_1 = u + v_0\) and \(u_2 = u - v_0\), we will discover that \(a\) and \(b\) are independent of \(u\).

\begin{lemma}
when \(u_1 = u + v_0\) and \(u_2 = u - v_0\), we have
\[
\begin{align*}
(7) \quad & \frac{cn u (sn u, cn u_2 - sn u, cn u_1)}{sn u_1 - sn u_2} = cn v_0, \\
(8) \quad & \frac{sn u (sn u, cn u_2 - sn u, cn u_1)}{cn u_2 - cn u_1} = \frac{cn v_0}{dn v_0}.
\end{align*}
\]
\end{lemma}

\textbf{proof} In accordance with the addition theorem of elliptic functions, we have
\[
\begin{align*}
& cn u_2 - cn u_1 = cn(u - v_0) - cn(u + v_0) = \frac{2s_1 d_1 s_2 d_2}{1 - k^2 s_1^2 s_2^2}, \\
& sn u_1 - sn u_2 = sn(u + v_0) - sn(u - v_0) = \frac{2s_1 c_1 d_1}{1 - k^2 s_1^2 s_2^2}, \\
& sn u, cn u_2 = sn(u + v_0) cn(u - v_0) = \frac{s_1 c_2 d_2 + s_2 c_3 d_1}{1 - k^2 s_1^2 s_2^2}, \\
& sn u, sn u_1 = sn(u - v_0) cn(u + v_0) = \frac{s_1 c_2 d_2 - s_2 c_3 d_1}{1 - k^2 s_1^2 s_2^2},
\end{align*}
\]

(Cayley [1] pp.65-66) , where we used abbreviations \(s_1 = sn u, c_1 = cn u, d_1 = dn u, s_2 = sn v_0, c_2 = cn v_0, d_2 = dn v_0\). Substituting these into the left hand sides of (7) and (8), we have the equalities (7) and (8).

Conversely, let us \(a = Rc v_0\) and \(b = \frac{Rc v_0}{dn v_0}\). Writing simply \(c_2 = cn v_0, d_2 = dn v_0\)
\[
s_1 = sn u, c_1 = cn u , \text{ we have } a = Rc_2, \quad b = \frac{Rc_2}{d_2}\text{ and hence the tangent line (2) is }
\]
(9) \(c_1x + d_1s_1y = Rc_2\).

We are going to find the intersection points of (9) with the circle

(10) \(x^2 + y^2 = R^2\).

From (9) and (10) killing \(x\), we have

(11) \((d_2^2s_1^2 + c_1^2)y^2 - 2Rc_2d_1s_1y + R^2(c_2^2 - c_1^2) = 0\).

Again we can use the following equations from [1]p.65,

(12) \(sn(u + v_o) + sn(u - v_o) = \frac{2s_1c_2d_1}{1 - k^2s_1^2s_2^2}\)

and

(13) \(sn(u + v_o)sn(u - v_o) = \frac{s_1^2 - s_2^2}{1 - k^2s_1^2s_2^2} \).

Noticing \((d_2^2s_1^2 + c_1^2) = 1 - k^2s_1^2s_2^2\) and \(c_2^2 - c_1^2 = s_1^2 - s_2^2\), we have

(14) \((d_2^2s_1^2 + c_1^2)y^2 - 2Rc_2d_1s_1y + R^2(c_2^2 - c_1^2)\)

\[= (d_2^2s_1^2 + c_1^2)\left(y^2 - \frac{2Rc_2d_1y}{1 - k^2s_1^2s_2^2} + R^2\frac{s_1^2 - s_2^2}{1 - k^2s_1^2s_2^2}\right)\]

\[= (d_2^2s_1^2 + c_1^2)\left(y - Rsn(u + v_o)\right)^2\left(y - Rsn(u - v_o)\right)\],

whence from (10) and (11), \(y = Rsn(u + v_o), \ x = Rcn(u + v_o) \) or \(y = Rsn(u - v_o)\)

\(x = Rcn(u - v_o)\). We have thus two intersection points: \(P_1 = (Rcn(u + v_o), Rsn(u + v_o))\)

and \(P_2 = (Rcn(u - v_o), Rsn(u - v_o))\). Thus we conclude that if there exists a circle

\(x^2 + y^2 = R^2\) which contains the ellipse \(x^2 + d_2^2y^2 = c_2^2R^2\) \((d_2 = dnv_o, \ c_2 = cnv_o)\), and

if the line combining \(P_1 \) and \(P_2\) is tangent at the point \(Q\), then we have \(P_1 = P(u + v_o),\)

\(P_2 = P(u - v_o)\) and \(Q = Q(u)\) depicted in the following picture.
In other words, if \( a = Rcn v_0 \) and \( b = \frac{Rcn v_0}{dn v_0} \), we have the tangent lines through points \( Q = Q(u) \) on the ellipse, with the segment \( P_1 \rightarrow P_2 \) whose parameter differs just \( 2v_0 \) for whichever values of \( u \) are. We have now the Poncelet’s closure theorem.

**Theorem** \( \) Let \( n \geq 3 \) be a natural number. Suppose there exists a circle \( x^2 + y^2 = R^2 \) which contains the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) \( \) (\( a = Rcn v_0 \), \( b = \frac{Rcn v_0}{dn v_0} \)). If we make tangents to the ellipse starting from a point \( P = P_1 \) on the circle, and get another point \( P_2 \) on the circle. Repeating the process from \( P_2 \), we have again \( P_3 \) by making tangent to the ellipse. In this way we have a sequence \( P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow \cdots \rightarrow P_m \) and we suppose \( P_{n+1} = P_1 \). In this case we must obtain \( P_1' \rightarrow P_2' \rightarrow P_3' \rightarrow \cdots \rightarrow P'_{m} \) and \( P'_{n+1} = P_1' \) even from any other starting point \( P = P_1' \) on the circle.

Proof In order to get \( P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow \cdots \rightarrow P_n \) and \( P_{n+1} = P_1 \), we must have \( 2v_0 = m \cdot 4K \) for some integer \( m \). This condition does not depend on the abscissa of the starting point \( P_1 \).

[2] Alfred George Greenhill, the Application of Elliptic Functions, Merchant Books