A GENERAL SOLUTION OF A PROBLEM IN LINEAR PREDICTION OF STATIONARY PROCESSES

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1. Statement of the Problem

Let $\zeta(t)$ and $\eta(t)$, $-\infty < t < \infty$, be processes which are stationary in the wide sense and stationarily dependent. We shall denote by $H = H_\zeta + H_\eta$ the Hilbert space of values of the processes $\zeta(t)$ and $\eta(t)$, where

$$H_\zeta = \bigoplus_{-\infty}^{\infty} \zeta(t), \quad H_\eta = \bigoplus_{-\infty}^{\infty} \eta(t).$$

The scalar product of vectors $f$, $g$ in $H$, belonging to the linear span of the variables $\zeta(t)$ and $\eta(t)$, is defined by the formula

$$(f, g) = \mathbb{E}\{f \tilde{g}\}.$$

This note deals with the solution of the following problem in prediction theory: from known values of the variables $\zeta(t)$, $t \leq 0$, and $\eta(t)$, $t \geq T$, find the best linear prediction of the variable $\zeta(\tau)$, $\tau > 0$. In other words, it is required to find the projection of the vector $\zeta(\tau) \in H$ onto the subspace

$$\mathcal{O} = H_\zeta^{-}(0) + H_\eta^{+}(T),$$

where $H_\zeta^{-}(0) = \bigoplus_{-\infty}^{0} \zeta(t)$, $H_\eta^{+}(T) = \bigoplus_{T}^{\infty} \eta(t)$.

Note that in the particular case when the processes $\zeta(t)$ and $\eta(t)$ differ with probability zero, and $T > 0$, the problem reduces to linear interpolation of the unknown values of $\zeta(\tau)$, $0 < \tau < T$, that is, to finding the projection of the vectors $\zeta(\tau)$, $0 < \tau < T$, onto the subspace

$$\mathcal{O} = H_\zeta^{-}(0) + H_\zeta^{+}(T),$$

where

$$H_\zeta^{+}(T) = \bigoplus_{T}^{\infty} \zeta(t).$$

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1 The linear closed span of vectors belonging to the sets $\mathcal{O}_i$ is denoted by the symbol $\bigvee_i \mathcal{O}_i$. 394
As much as we know, the solution of this problem even for a process with continuous time was until now known only for the case when the spectral density of the process is rational (see [1]-[4]).

We assume below that time $t$ is continuous, and that the correlation functions of the processes under consideration are continuous. For simplicity we shall restrict ourselves to one-dimensional processes $\xi(t)$ and $\eta(t)$. The presentation is carried out in such a way that our arguments and results can be carried over without any complications (with appropriate changes) to multidimensional processes of maximal rank with continuous and discrete time.

All the definitions and results of the theory of stationary random processes, which are considered known in our work, can be found in [2].

2. Adopted Assumptions

The stationary process $\xi(t)$ can be represented, and moreover in a unique manner, in the form $\xi(t) = \xi'(t) + \xi''(t)$, where the processes $\xi'(t)$ and $\xi''(t)$ are subordinate to the process $\xi(t)$ and are uncorrelated, the process $\xi'(t)$ being linearly regular while $\xi''(t)$ is linearly singular. In what follows it is assumed that the process $\xi(t)$ is linearly regular, i.e., it is assumed that $\xi(t) = \xi'(t)$. The generality of the analysis is practically not violated, since the process $\xi''(t)$ is exactly extrapolated only from values of the process $\xi(t)$ for $t \leq 0$.

In the space $H$ of values of the processes $\xi(t)$ and $\eta(t)$ a continuous group of unitary operators $U_t$ can be found such that

$$
\xi(t) = U_t \xi(0) = \int_{-\infty}^{\infty} e^{i\lambda t} dE_\lambda \xi(0),
$$

(1)

$$
\eta(t) = U_t \eta(0) = \int_{-\infty}^{\infty} e^{i\lambda t} dE_\lambda \eta(0),
$$

where $E_\lambda$ is the spectral function of the group $U_t$. The subspaces $H_\xi$ and $H_\eta$ reduce the group $U_t$.

The space $H$ is representable in the form $H_a \oplus H_s$, where $H_a$ is the set of all the vectors $f$ of $H$ for which the spectral function $(E_\lambda f, f)$ is absolutely continuous, $H_s$ is the set of all the vectors of $H$ for which the spectral function is singular; the sets $H_a$ and $H_s$ are subspaces reducing the group $U_t$ [5]. In view of the linear regularity of the process $\xi(t)$, its spectral function $(E_\lambda \xi(0), \xi(0))$ is absolutely continuous, and therefore $\xi(t) \in H_a$. Let $\eta_a(0)$ and $\eta_s(0)$ be projections of the vector $\eta(0)$ onto the subspaces $H_a$ and $H_s$, respectively,

$$
\eta_a(t) = U_t \eta_a(0) \quad \text{and} \quad \eta_s(t) = U_t \eta_s(0).
$$

Then $\eta(t) = \eta_a(t) + \eta_s(t)$; $\eta_a(t)$ and $\eta_s(t)$ are stationary processes, where $\eta_a(t) \in H_a$, $\eta_s(t) \in H_s$, and the processes $\eta_s(t)$ and $\xi(t)$ are uncorrelated, since $\xi(t) \in H_a$. Since the information received from observations of the process $\eta_s(t)$ cannot be used for a linear prediction of the variables $\xi(t)$, we shall assume
that $\eta_a(t) = 0$, i.e., that the spectral function of the process $\eta(t)$ is absolutely continuous.

We shall denote by $f_{\xi(t)}(\lambda)$ and $f_{\eta(t)}(\lambda)$ the spectral densities of the processes $\xi(t)$ and $\eta(t)$, and by $f_{\xi\eta}(\lambda)$ their cross-density, $f_{\xi\eta}(\lambda) = d(E\xi(0), \eta(0))/d\lambda$. In spite of the fact that for a unique definition of the latter it is not enough in general to know the variables $\xi(t)$ for $t \leq 0$ and $\eta(t)$ for $t \geq T$, the function $f_{\xi\eta}(\lambda)$ is assumed known.

3. Solution of the Problem in the Case when the Process $\eta(t)$ is Linearly Singular

Let $H_\eta^\perp = H \ominus H_\eta$, $\xi_\eta(0)$ and $\xi_\eta^\perp(0)$ be the projections of the vector $\xi(0)$ onto $H_\eta$ and $H_\eta^\perp$, respectively. Since the subspaces $H_\eta$ and $H_\eta^\perp$ reduce the group $U_\eta$,

$$
\xi_\eta(t) \equiv U_{\eta} \xi_\eta(0) \in H_\eta, \quad \xi_\eta^\perp(t) \equiv U_{\eta} \xi_\eta^\perp(0) \in H_\eta^\perp.
$$

We shall represent the process $\xi(t)$ uniquely in the form

$$
\xi(t) = \xi_\eta(t) + \xi_\eta^\perp(t),
$$

where the processes $\xi_\eta(t)$ and $\xi_\eta^\perp(t)$ are uncorrelated, and $\xi_\eta(t) \in H_\eta = H_\eta^\perp(T) \subset H_\eta(0) + H_\eta^\perp(T)$: hence the process $\xi_\eta(t)$ is extrapolated exactly. Thus, in this case it is actually required only to predict the behavior of the stationary process $\xi_\eta^\perp(t)$ from its values for $t \leq 0$.

It is easy to see that the process $\xi_\eta(t)$ is obtained from the process $\eta(t)$ by the linear transformation

$$
\xi_\eta(t) = \int_{-\infty}^{\infty} \varphi_{\xi\eta}(\lambda) dE_{\lambda}\eta(0),
$$

where

$$
\varphi_{\xi\eta}(\lambda) = f_{\xi\eta}(\lambda)/f_{\eta}(\lambda) \quad \text{if} \quad f_{\eta}(\lambda) \neq 0 \quad \text{and} \quad \varphi_{\xi\eta}(\lambda) = 0 \quad \text{if} \quad f_{\eta}(\lambda) = 0.
$$

By virtue of (1), (2) and (3) the spectral density $f_{\xi_\eta}(\lambda)$ of the process $\xi_\eta^\perp(t)$ is equal to

$$
\Sigma_{\xi\eta}(\lambda) f_{\xi(\lambda)}(\lambda), \quad \text{where} \quad \Sigma_{\xi\eta}(\lambda) = 1 - |f_{\xi(\lambda)}(\lambda)|^2 f_{\eta}(\lambda) f_{\eta}(\lambda),
$$

if

$$
f_{\eta}(\lambda) \neq 0 \quad \text{and} \quad \Sigma_{\xi\eta}(\lambda) = 1 \quad \text{if} \quad f_{\eta}(\lambda) = 0.
$$

Noting that

$$
\int_{-\infty}^{\infty} \log f_{\xi\eta}(\lambda) \frac{d\lambda}{1 + \lambda^2} > -\infty,
$$

we arrive, by virtue of the assumed linear regularity of the process $\xi(t)$ and using a known formula giving the linear extrapolation (see [2]), at the following conclusion.
In the case under consideration the condition

$$\int_{-\infty}^{\infty} \frac{\log \Sigma \xi(t)}{1 + \lambda^2} d\lambda = -\infty$$

is necessary and sufficient for the exact extrapolation of the process $\xi(t)$, and hence in view of (2) of the process $\xi(t)$ as well.

But if

$$\int_{-\infty}^{\infty} \frac{\log \Sigma \xi(t)}{1 + \lambda^2} d\lambda > -\infty,$$

then the best linear prediction $\xi(t)$ of the process $\xi(t)$ for the time instant $t > 0$ (the projection of the vector $\xi(t)$ onto the subspace $H_{\xi}(0 - H_{\xi}(T))$) is given by the expression

$$\xi(t) = \sum_{-\infty}^{\infty} \psi(t) \left\{ \int_{0}^{\tau} e^{-i\lambda \xi(s)} ds \right\} dE(t)$$

where $\Gamma(\lambda)$ is the boundary value of the function

$$\Gamma(\lambda) = \sqrt{\pi} \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \log f(\lambda) \left( \frac{1}{1 + \lambda^2} \right) d\lambda \right\}$$

which is analytic in the lower half-plane, and

$$c(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda s} \Gamma(\lambda) d\lambda.$$

4. The Transfer Function $S(\lambda)$

Now let the process $\eta(t)$ be linearly regular.

We shall denote by $L_2$ the Hilbert space of measurable functions $f(\lambda)$ defined on the real line $(-\infty, \infty)$, for which

$$\|f\|_2^2 = \int_{-\infty}^{\infty} \frac{1}{\pi} f(\lambda) \frac{d\lambda}{1 + \lambda^2} < \infty.$$

In $L_2$ we shall reduce the subspaces $H_{\xi+}^2$ and $H_{\xi-}^2$ of boundary values of functions $f(z)$ which are analytic in the half-planes $\text{Im } z > 0$ and $\text{Im } z < 0$ and satisfy the condition

$$\sup_{\tau > 0} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} |f(\lambda \pm i\tau)|^2 d\lambda \right\} < \infty.$$
in the half-planes corresponding to them. The subspaces $H^+_2$ and $H^-_2$ are orthogonal, and $L_2 = H^+_2 \oplus H^-_2$.

Let $\Gamma_\xi(\lambda)$ and $\Gamma_\eta(\lambda)$ be functions in $H^-_2$ which are boundary values of the functions $\Gamma_\xi(z)$ and $\Gamma_\eta(z)$ given by a formula of type (6) from the spectral densities $f_\xi(\omega)$ and $f_\eta(\omega)$. The functions $\Gamma_\xi(\lambda)$ and $\Gamma_\eta(\lambda)$ factor the densities $f_\xi(\omega)$ and $f_\eta(\omega)$, respectively,

$$\pi f_\xi(\omega) = |\Gamma_\xi(\lambda)|^2, \quad \pi f_\eta(\omega) = |\Gamma_\eta(\lambda)|^2,$$

and belong to the class of functions $\Gamma(\lambda)$ of $H^-_2$ having the property

$$\left\{ e^{i\beta t} \Gamma(\lambda) \right\} = H^-_2.$$

Note that for each such function $\Gamma(\lambda)$,

$$\Gamma(\lambda) \in H^+_2, \quad \left\{ e^{i\beta t} \Gamma(\lambda) \right\} = H^+_2.$$

We shall denote by $F_\xi$ the isometric mapping of the subspace $H_\xi$ into $L_2$, operating on the set of vectors of the form $f = \sum_{j=1}^n \alpha_j \xi(t_j)$ (the $\alpha_j$ are complex numbers) which is dense in $H_\xi$ according to the formula

$$F_\xi \left( \sum_{j=1}^n \alpha_j \xi(t_j) \right) = \sum_{j=1}^n \alpha_j e^{i\beta t_j} \Gamma(\lambda).$$

By virtue of (9) and (7), the operator $F_\xi$ has the following properties:

$$F_\xi U_\xi f(\lambda) = e^{i\beta t} (F_\xi f)(\lambda),$$

$$F_\xi H_\xi(0) = H^-_2, \quad F_\xi H_\xi = L_2.$$  
(10.a)

From (10.a) it also follows that

$$F_\xi E(\Delta) f(\lambda) = \chi_\Delta(\lambda) (F_\xi f)(\lambda),$$

where $\chi_\Delta(\lambda)$ is the characteristic function of the interval $\Delta$.

We shall denote by $F_\eta$ the isometric mapping of the space $H_\eta$ onto $L_2$, differing from $F_\xi$, besides the notation, by the fact that

$$F_\eta H_\eta(t) = e^{i\beta t} \Gamma(\lambda),$$

and hence

$$F_\eta H_\eta^+(T) = H^+_2.$$  
(13)

We shall extend the operators $F_\xi$ and $F_\eta$ to the entire space $H$, defining them to be zero on the subspaces $H^\perp_\xi = H \ominus H_\xi$ and $H^\perp_\eta = H \ominus H_\eta$, respectively. We shall preserve the previous notations for the extended operators. We point out the following obvious relations:

$$F_\xi^* F_\xi = P_\xi, \quad F_\eta^* F_\eta = P_\eta, \quad F_\xi F_\eta^* = I, \quad F_\eta F_\xi^* = I,$$

where $P_\xi$ and $P_\eta$ are projection operators onto the subspaces $H_\xi$ and $H_\eta$, and $I$ is the identity operator in the space $L_2$. 

(14)
We shall introduce into the consideration the operator

\[ S_{\xi\eta} = F_\xi F_\eta^*, \]

mapping the space \( L_2 \) into itself. Since \( \|S_{\xi\eta}\| \leq 1 \), and since, by virtue of (10),

\[ (S_{\xi\eta} e^{it\lambda}f(\cdot))(\lambda) = e^{it\lambda}(S_{\xi\eta} f(\cdot))(\lambda), \]

there exists a measurable function \( S_{\xi\eta}(\lambda) \) such that (see [6])

\[ (S_{\xi\eta} f)(\lambda) = S_{\xi\eta}(\lambda) f(\lambda), \quad \text{ess sup}_{\lambda} |S_{\xi\eta}(\lambda)| = \| S_{\xi\eta} \| \leq 1. \]

We shall express the function \( S_{\xi\eta}(\lambda) \) in terms of the spectral densities \( f_\xi(\lambda), f_{\eta\eta}(\lambda) \) and \( f_{\xi\eta}(\lambda) \). By virtue of the properties (11) and (14) of the mappings \( F_\xi \) and \( F_\eta \), for any interval \( \Delta \) of the real line we have

\[
\begin{align*}
(E(\Delta)\xi(0), \eta(0)) &= \frac{1}{\pi} \int_{-\infty}^{\infty} (F_\xi E(\Delta)\xi(0)) \cdot (F_\eta \eta(0))(\lambda) \, d\lambda \\
&= \frac{1}{\pi} \int_{\Delta} (F_\xi \xi(0))(\lambda) \cdot (F_\eta \eta(0))(\lambda) \, d\lambda \\
&= \frac{1}{\pi} \int_{\Delta} S_{\xi\eta}(\lambda) e^{i\lambda T} \Gamma_\xi(\lambda) \Gamma_\eta(\lambda) \, d\lambda.
\end{align*}
\]

On the other hand, for any interval \( \Delta \) the equation

\[ (E(\Delta)\xi(0), \eta(0)) = \int_{\Delta} \frac{d(E_\xi \xi(0), \eta(0))}{d\lambda} \, d\lambda = \int_{\Delta} f_{\xi\eta}(\lambda) \, d\lambda \]

holds. By comparing (17) and (18), we obtain

\[ S_{\xi\eta}(\lambda) = \pi e^{i\lambda T} f_{\eta\eta}(\lambda)/\Gamma_\xi(\lambda) \Gamma_\eta(\lambda). \]

In particular, for the usual interpolation problem, where \( \xi(t) = \eta(t) \),

\[ f_{\eta\eta}(\lambda) = f_{\xi\xi}(\lambda) = \frac{1}{\pi} |\Gamma_\xi(\lambda)|^2, \quad T > 0, \]

the function \( S_{\xi\xi}(\lambda) \) is computed from the formula

\[ S_{\xi\xi}(\lambda) = e^{i\lambda T} \Gamma_\xi(\lambda)/\Gamma_\xi(\lambda). \]

As will be seen from what follows, the function \( S_{\xi\eta}(\lambda) \) plays a primary role in the solution of the problem formulated in § 1.

5. The Functional Model

Along with \( S_{\xi\eta}(\lambda) \) we shall introduce into the analysis the function

\[ \Sigma_{\xi\eta}(\lambda) = 1 - |S_{\xi\eta}(\lambda)|^2 = 1 - |f_{\xi\eta}(\lambda)|^2/f_{\xi\xi}(\lambda) f_{\eta\eta}(\lambda). \]

Obviously, \( \Sigma_{\xi\eta}(\lambda) \geq 0 \). We shall denote by \( L_2^{\Sigma} \) the subspace of \( L_2 \) which is the closure of the domain of values of the product operator on the function \( \Sigma_{\xi\eta}^{1/2}(\lambda) \). We shall show that
part of the group of unitary operators $U_t$ on the subspace $H^1_\xi$ reducing it is isomorphic to the group of product operators on $e^{i\lambda T}$ in $L^2_\xi$.

In fact, if $P_\xi^1$ is the orthogonal projection operator of $H$ onto the subspace $H^1_\xi$, then

$$H^1_\xi = P_\xi^1(H^1_\xi + H_\eta) \subseteq \overline{P_\xi^1(H_\xi + H_\eta)} = \overline{P_\xi^1 H_\eta} \subseteq H^1_\xi$$

and hence, $H^1 \subseteq \overline{P_\xi^1 H_\eta}$.

Let $f = P_\xi^1 h$, where $h \in H_\eta$. By virtue of (14) and (15),

$$\|f\|^2 = (P_\xi^1 h, h) = \|h\|^2 - (P_\xi^1 h, h)$$

(21) 

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} |(F_\eta h)(\lambda)|^2 \, d\lambda - \frac{1}{\pi} \int_{-\infty}^{\infty} |(F_\eta F_\eta^* F_\eta h)(\lambda)|^2 \, d\lambda$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \left| \sum_{n=1}^{1/2}(\lambda)(F_\eta h)(\lambda) \right|^2 \, d\lambda.$$

If to each vector of the form $f = P_\xi^1 h$, $h \in H_\eta$, there is put in correspondence the function $\sum_{n=1}^{1/2}(\lambda)(F_\eta h)(\lambda)$, then as follows from (21) the mapping thus obtained will be isometric and have a dense domain of values in $L^2_\xi$. By extending this mapping by continuity to the entire $H^1_\xi$, we arrive at an isometric operator $F_\xi$ of $H^1_\xi$ on $L^2_\xi$. Since the subspace $H^1_\xi$ reduces the group $U_t$,

$$F_\xi(U_t P_\xi^1 h)(\lambda) = (P_\xi^1 P_\xi^1 U_t h)(\lambda)$$

$$= \sum_{n=1}^{1/2}(\lambda) F_\eta U_t h)(\lambda) = e^{i\lambda t} \sum_{n=1}^{1/2}(\lambda)(F_\eta h)(\lambda), \quad h \in H_\eta,$$

and hence $(F_\xi U_t f) = e^{i\lambda t}(F_\xi f)(\lambda)$ for any $f \in H^1_\xi$. The assertion is proved.

Let us extend the operator $F_\xi$ to the entire space $H$, by setting $F_\xi^1 = 0$, $h \in H_\xi$, and preserve the previous notation for the extended operator. The operator $F = F_\xi + F_\xi^1$ maps $H$ onto the space of functions $L^2_\xi = L^2_2 \oplus L^2_\xi$ and has the following property:

$$(F U_t f)(\lambda) = e^{i\lambda t}(F f)(\lambda), \quad f \in H.$$

Each function $f(\lambda) \in L^2_\xi$ can be represented in the form $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$, where $f_1(\lambda) \in L_2$ and $f_2(\lambda) \in L^2_\xi$. It is obvious that

$$FH_\xi = L_2 \oplus \{0\}, \quad FH^-_\xi = H^-_2 \oplus \{0\}.$$

For $f \in H_\eta$ we have

$$(FP_\xi f)(\lambda) = (F_\xi f)(\lambda) \oplus \{0\} = (F_\xi F_\eta^* F_\eta f)(\lambda) \oplus \{0\}$$

$$= S_{\eta}(\lambda)(F_\eta f)(\lambda) \oplus \{0\} ,$$

$$(FP_\xi f)(\lambda) = \{0\} \oplus (F_\xi P_\xi^1 f)(\lambda) = \{0\} \oplus \sum_{n=1}^{1/2}(\lambda)(F_\eta f)(\lambda),$$
and hence,

\[(Ff)(\lambda) = S_{\xi}(\lambda)(F_\eta f)(\lambda) \oplus \Sigma_{\xi}^{1/2}(\lambda)(F_\eta f)(\lambda).\]

The arguments given prove the validity of the following proposition.

There exists an isometric operator \( F \) mapping the space of values of the processes \( \xi(t) \) and \( \eta(t) \) onto the space of functions \( L^2_2 = L_2 \oplus L^2_2 \) and having the following properties:

\[ (FU, f)(\lambda) = e^{i\lambda t}(Ff)(\lambda), \quad f \in H, \]

\[ (FH_\xi)(0) = H_2 \oplus \{0\}, \]

\[ FH_\eta^+(T) = \{S_{\xi}(\lambda)f(\lambda) \oplus \Sigma_{\xi}^{1/2}(\lambda)f(\lambda), f(\lambda) \in H_2^+\}. \]

In particular,

\[ (F\xi(t))(\lambda) = e^{i\lambda t}I_\xi(\lambda) \oplus \{0\}, \]

\[ (F\eta(t))(\lambda) = e^{i\lambda t}S_{\xi}(\lambda) e^{-i\lambda T}I_\eta(\lambda) \oplus e^{i\lambda t} \Sigma_{\xi}^{1/2}(\lambda) e^{-i\lambda T}I_\eta(\lambda). \]

A more general version of this proposition has been proved in the authors’ paper [7].

By virtue of what was stated above, in order to find the best linear prediction of the process \( \xi(t) \) from the values of the variables \( \xi(t) \) for \( t \leq 0 \) and \( \eta(t) \) for \( t \geq T \), one can project the function \( e^{i\lambda t}I_\xi(\lambda) \oplus \{0\} \) onto the subspace \( H_\xi^-(0) + H_\eta^+(T) \), where \( H_\xi^-(0) = FH_\xi(0) \) and \( H_\eta^+(T) = FH_\eta^+(T) \), and then transform the function so obtained by the operator \( F^* \) into the original space \( H \).

Note that on the set of functions representable in the form

\[ f(\lambda) = [f_1(\lambda) + S_{\xi}(\lambda)f_2(\lambda)] \oplus \Sigma_{\xi}^{1/2}(\lambda)f_2(\lambda), \]

where \( f_1(\lambda), f_2(\lambda) \in L_2 \), which is dense in \( L^2_2 \) (on the image of the linear manifold \( H_\xi + H_\eta \)), the operator \( F^* \) acts according to the formula

\[ F^*f(\cdot) = \int_{-\infty}^{\infty} f_1(\lambda) \frac{dE_2\xi(\lambda)}{\Gamma_\xi(\lambda)} + \int_{-\infty}^{\infty} e^{i\lambda T} f_2(\lambda) \frac{dE_2\eta(\lambda)}{\Gamma_\eta(\lambda)}. \]

6. Projection onto \( \tilde{H}_\xi^-(0) \) and \( \tilde{H}_\eta^+(T) \)

Let us find the projections \( \tilde{F}_\xi f \) and \( \tilde{F}_\eta^+ f \) of an arbitrary function \( f(\lambda) = f_1(\lambda) \oplus f_2(\lambda) \) of \( L^2_2 \) onto the subspaces \( \tilde{H}_\xi^-(0) \) and \( \tilde{H}_\eta^+(T) \). We shall denote by \( \pi_\pm \) the projection operators onto the subspaces \( H_2^\pm \) in \( L_2 \),

\[ (\pi_\pm f)(\lambda) = \pm \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda' - \lambda \mp i\varepsilon} d\lambda'. \]
On the basis of (21) the following formula holds for the projection operator $\mathcal{P}_\xi^-$:

\[(\mathcal{P}_\xi^- f)(\lambda) = (\pi^- f_1)(\lambda) \oplus \{0\}.\] (25)

Since for the function $g(\lambda) = (g_1(\lambda) \oplus g_2(\lambda)) \in L^S_2 \ominus \mathcal{H}_n^+(T)$ and any function $\phi(\lambda) \in H_2^+$, the equation

\[\frac{1}{\pi} \int_{-\infty}^{\infty} [S_{\xi\eta}(\lambda)g_1(\lambda) + \Sigma^{1/2}_{\xi\eta}(\lambda)g_2(\lambda)]\phi(\lambda) \, d\lambda = 0\]

holds by virtue of (22), each function $g(\lambda) \in L^S_2 \ominus \mathcal{H}_n^+(T)$ satisfies the condition

\[(26) \quad [\pi_+(S_{\xi\eta}g_1 + \Sigma^{1/2}_{\xi\eta}g_2)](\lambda) = 0.\]

For an arbitrary function $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda)$, one can write

\[f_1(\lambda) = S_{\xi\eta}(\lambda)h(\lambda) + g_1(\lambda),\] (27)

\[f_2(\lambda) = \Sigma^{1/2}_{\xi\eta}(\lambda)h(\lambda) + g_2(\lambda),\]

where $h(\lambda) \in H_2^+$ and $S_{\xi\eta}(\lambda)h(\lambda) \oplus \Sigma^{1/2}_{\xi\eta}(\lambda)h(\lambda)$ is the projection of the function $f(\lambda)$ onto $H_n^+(T)$, and $g_1(\lambda) \oplus g_2(\lambda) \in L^S_2 \ominus \mathcal{H}_n^+(T)$. Multiplying the first equation in (27) by $S_{\xi\eta}f_1$ and the second by $\Sigma^{1/2}_{\xi\eta}f_2$, then adding the rows and applying the operator $\pi_+$, we obtain, on the basis of (26),

\[(28) \quad h(\lambda) = [\pi_+(S_{\xi\eta}f_1 + \Sigma^{1/2}_{\xi\eta}f_2)](\lambda).\]

From (27) and (28) it follows that

\[\mathcal{P}_\xi^+ f = S_{\xi\eta}(\lambda)[\pi_+(S_{\xi\eta}f_1 + \Sigma^{1/2}_{\xi\eta}f_2)](\lambda)\]

\[\oplus \Sigma^{1/2}_{\xi\eta}(\lambda)[\pi_+(S_{\xi\eta}f_1 + \Sigma^{1/2}_{\xi\eta}f_2)](\lambda).\] (29)

7. Projection onto the Closure of the Sum of Subspaces

In order to find the projection of any function in $L^S_2$ onto the subspace $\mathcal{H}_n^-(0) \ominus \mathcal{H}_n^+(T)$, we use the following quite general geometric arguments.
Let $H_1$ and $H_2$ be subspaces of the Hilbert space $H, \mathcal{K} = H_1 + H_2$; $P_1, P_2$ and $P_K$ are projection operators (orthogonal) onto $H_1, H_2$ and $K$. Then the projection operator $P_K^l = I - P_K$ onto the subspace $K^\perp = H \ominus K$ can be found from the formula

\begin{equation}
(30) \quad P_K^l = s\lim_{n \to \infty} Q^n, \quad Q = P_1^l P_2^l P_1^l,
\end{equation}

where $P_1^l, P_2^l$ are projections onto $H_1^\perp = H \ominus H_1$ and $H_2^\perp = H \ominus H_2$, and $s\lim$ signifies the strong limit.

In fact, as is seen from (30), $Q$ is a non-negative self-adjoint operator, and $\|Q\| \leq 1$. The monotonically decreasing sequence of self-adjoint operators $\{Q^n\}$ is bounded from below by zero and, therefore, converges strongly (see [5]) to some positive self-adjoint operator $P'$. Since

$$Q P' = Q \cdot s\lim_{n \to \infty} Q^n = s\lim_{n \to \infty} Q^{n+1} = P', \quad (P')^2 = s\lim_{n \to \infty} Q^n P' = P',$$

and for any $f$ of the subspace $K' = \{ f : Qf = f \}$

$$P' f = s\lim_{n \to \infty} Q^n f = f,$$

the operator $P'$ is an orthogonal projection operator onto the subspace $K'$. From the fact that $K^\perp = H_1^\perp \cap H_2^\perp$ follows the inclusion $K^\perp \subset K'$. On the other hand, for $f \in K'$ the equations

$$P_1 f = P_1 Q f = 0,$$

$$\|P_2 f\|^2 = \|f\|^2 - \|P_1^l P_2^l f\|^2 \leq \|f\|^2 - \|P_1^l (P_1 + P_2^l) f\|^2 = \|f\|^2 - \|Q f\|^2 = 0$$

hold which are equivalent to the inclusion $K' \subset K^\perp$. Thus $K' = K^\perp$ and $P' = P_K^l$.

Since

$$(I - P_2^l P_1^l) \sum_{k=0}^{n} (P_2^l P_1^l)^k = I - (P_2^l P_1^l)^{n+1} = I - P_2^l Q^n$$

and $I - P_2^l P_1^l = P_1 + P_2 P_1^l$,

\begin{equation}
(31) \quad I - P_2^l Q^n = (P_1 + P_2 P_1^l) \sum_{k=0}^{n} (P_2^l P_1^l)^k,
\end{equation}

and, therefore

$$P_K = I - s\lim_{n \to \infty} Q^n = I - P_2^l s\lim_{n \to \infty} Q^n$$

$$= s\lim_{n \to \infty} (I - P_2^l Q^n) = s\lim_{n \to \infty} (P_1^l + P_2 P_1^l) \sum_{k=1}^{n} (P_1 P_2)^k.$$

Hence, for any $f \in H$ a sequence of approximations $f_n$ in $K$ can be found,

$$f_n = (P_1 + P_2 P_1^l) \sum_{k=0}^{n} (P_2^l P_1^l)^k f,$$

converging to the vector $P_K f$ as $n \to \infty$. 
The errors in these approximations $\delta_n(f) = \|f - f_n\|$ form a monotonically decreasing sequence. In fact, in view of (30) and (31),

$$\delta_n(f) = \|P_{1/2}Q^n f\| \geq \|P_{1/2}P_{1/2}P_{1/2}Q^n f\| = \|P_{1/2}Q^{n+1} f\| = \delta_{n+1}(f).$$

If for all vectors $f$ of some subspace $L \subset H$ the relation

$$\lim_{n \rightarrow \infty} \delta_n(f) = \|P_L f\| = 0$$

holds, then for each $g \in K^\perp$ the equation $P_L g = 0$ holds, where $P_L$ is the projection operator on $L$. Since $K^\perp = \{ f : f \in H^1_n, n \rightarrow \infty \} \subset L$, the inclusion $L \subset K$ is possible if and only if the equation $P_L f = 0$ does not have solutions in $H$ satisfying the conditions $f \in H^1$ and $\|P_L f\| > 0$.

8. The General Solution of the Problem in the Case when the Process $\eta(t)$ is Linearly Regular

The following proposition follows directly from the arguments given and from formulas (25), (29) and (24).

Let $\xi(t)$ and $\eta(t)$ be linearly regular stationarily dependent stationary processes, whose spectral representations are

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dE_{\lambda} \xi(0), \quad \eta(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dE_{\lambda} \eta(0),$$

let $f_{\xi}(\lambda)$ and $f_{\eta}(\lambda)$ be their spectral densities, and $f_{\xi\eta}(\lambda)$ their cross-spectral density. For an exact prediction of the process $\xi(t)$ from its values for $t \leq 0$ and the values of the process $\eta(t)$ for $t \geq T$, it is necessary and sufficient that the set of solutions $f(\lambda)$ in $L_2$ of the equation

$$S_{\xi\eta}(\lambda)(\pi_+ f)(\lambda) + (\pi_- f)(\lambda) = (1 - |S_{\xi\eta}(\lambda)|^2)^{1/2} g(\lambda),$$

where $\pi_+$ are projection operators defined in § 6, $S_{\xi\eta}(\lambda) = \pi \frac{e^{i\lambda T}f_{\eta}(\lambda)}{\Gamma_{\xi}(\lambda)} r_{\xi}(\lambda)$, and the functions $\Gamma_{\xi}(\lambda)$ and $\Gamma_{\eta}(\lambda)$ are defined by formula (6) from the densities $f_{\xi}(\lambda)$ and $f_{\eta}(\lambda)$, belong to the subspace $H_2^\perp$ for all functions $g(\lambda) \in L_2$.

If in the presence of the stated information the process $\xi(t)$ remains indeterminate, then the best linear prediction $\tilde{\xi}(T)$ of the process $\xi(t)$ for the time instant $T > 0$ can be found by the method of successive approximations. To do this one must set into correspondence to the variable $\xi(t)$ the function

$$\tilde{\xi}(T, \lambda) = e^{i\lambda T} \Gamma_{\xi}(\lambda) \oplus \{0\}$$

of the space $L_2^S = L_2 \oplus (1 - |S_{\xi\eta}(\lambda)|^2)^{1/2} L_2$, then with the aid of the operator $G$ in $L_2^S$, operating on $f(\lambda) = f_1(\lambda) \oplus f_2(\lambda) \in L_2^S$ according to the formula

$$(Gf)(\lambda) = (\pi_+ f_1)(\lambda) - S_{\xi\eta}(\lambda)[\pi_+(S_{\xi\eta}^S \pi_+ f_1 + \Sigma_{\xi\eta}^{1/2} f_2)](\lambda)$$

$$\oplus - f_2(\lambda) - \Sigma_{\xi\eta}^{1/2}(\lambda)[\pi_+(S_{\xi\eta}^S \pi_+ f_1 + \Sigma_{\xi\eta}^{1/2} f_2)](\lambda),$$
construct the sequence of functions \( \varphi^{(n)}(\tau, \lambda) = \varphi_1^{(n)}(\tau, \lambda) \oplus \varphi_2^{(n)}(\tau, \lambda) \in L_2^r \),
\[
\varphi^{(n)}(\tau, \lambda) = \sum_{k=0}^{n} (G^n \xi(\tau, \cdot))(\lambda),
\]
and from the functions \( \varphi^{(n)}(\tau, \lambda) \) in the space of values of the processes \( \xi(t) \) and \( \eta(t) \) establish the sequence of vectors
\[
\xi^{(n)}(\tau) = \int_{-\infty}^{\infty} \frac{(\pi - \varphi_1^{(n)}(\tau, \cdot))(\lambda) \gamma_\xi(\lambda)}{\Gamma_\xi(\lambda)} dE_\xi(0)
\]
\[
+ \int_{-\infty}^{\infty} e^{i\lambda T} \frac{1}{\Gamma_\eta(\lambda)} \left[ \pi_+ S_{\xi\eta} \pi_+ \varphi_1^{(n)}(\tau, \cdot) + \Sigma_{\xi}^{1/2} \varphi_2^{(n)}(\tau, \cdot) \right](\lambda) dE_\eta(0).
\]

Then each of the vectors \( \xi^{(n)}(\tau) \) represents a linear prediction of \( \xi(\tau) \) for \( \tau > 0 \), the better the larger \( n \) is, since the corresponding errors
\[
\delta_n = \left\{ E[\xi^{(n)}(\tau) - \xi^{(n)}(\tau)]^2 \right\}^{1/2}
\]
form a monotonically decreasing sequence. The best linear prediction \( \hat{\xi}(\tau) \) is the limit in the mean square of the sequence \( \xi^{(n)}(\tau) \).

### 9. The Solution of the Problem for Equivalent Processes \( \xi(t) \) and \( \eta(t) \)

Let \( \xi(t) \) and \( \eta(t) \) be linearly regular stationarily dependent equivalent processes. The latter means that \( H_\xi = H_\eta \). From the definition of the function \( S_{\xi\eta}(\lambda) \) it follows directly that the condition \( |S_{\xi\eta}(\lambda)| = 1 \) (almost everywhere) is necessary and sufficient for \( H_\xi = H_\eta \). For equivalent processes the problem being considered is noticeably simplified.

Based on the previous proposition, one can assert that for such processes for an exact prediction of the process \( \xi(t) \) from its values for \( t \leq 0 \) and the values of the process \( \eta(t) \) for \( t \geq T \), it is necessary and sufficient that the function \( S_{\xi\eta}(\lambda) \) not be factorable in the form \( S_{\xi\eta}(\lambda) = \varphi_+(\lambda)/\varphi_-(\lambda) \), where \( \varphi_{\pm}(\lambda) \in H_2^{2} \).

If the process \( \xi(t) \) is indeterminate, then the best linear prediction of the variable \( \xi(\tau), \tau > 0 \), can be found by the method of successive approximations. To do this one must construct, starting from the function \( \tilde{\xi}(\tau, \lambda) = e^{i\lambda T} \Gamma_\xi(\lambda) \) with the aid of the operator \( \tilde{Q} \), operating according to the formula
\[
(\tilde{Q} f)(\lambda) = (\pi_+ S_{\xi\eta} \pi_+ f)(\lambda), \quad f \in L_2,
\]
the sequence of functions
\[
\varphi^{(n)}(\tau, \lambda) = (\tilde{Q}^n \tilde{\xi}(\tau, \cdot))(\lambda),
\]
and then from these functions establish in the space \( H_\xi \) the sequence of vectors
\[
\xi^{(n)}(\tau) = \int_{-\infty}^{\infty} \left[ e^{i\lambda T} - \frac{\varphi^{(n)}(\tau, \lambda)}{\Gamma_\xi(\lambda)} \right] dE_\xi(0).
\]
Each of the vectors $\zeta^{(n)}(\tau)$ represents a linear prediction of the process $\zeta(\tau)$ for $\tau > 0$, the better the larger $n$ is. The best linear prediction $\zeta(\tau)$ is the limit of the sequence of vectors $\zeta^{(n)}(\tau)$ as $n \to \infty$. Note that the last proposition contains a general method of solving the linear interpolation problem.

If the function $S_{\zeta\eta}(\lambda)$ is representable in the form

$$S_{\zeta\eta}(\lambda) = B_1(\lambda)/B_2(\lambda),$$

where $B_1(\lambda)$ and $B_2(\lambda)$ are boundary values of the functions $B_1(z)$ and $B_2(z)$ which are analytic in the upper half-plane and satisfy the condition

$$|B_i(z)| < 1, \quad \text{Im} z > 0; \quad |B_i(\lambda)| = 1 \quad (\text{a.e.}), \quad \text{Im} \lambda = 0, \quad i = 1, 2,$

where $B_2(z)$ is a finite Blaschke product, then the desired best linear prediction of the process $\zeta(t)$ can be found in finite form.

We shall show how to do this. Let us denote by $B_1$ and $B_2$ the unitary product operators on the functions $B_1(\lambda)$ and $B_2(\lambda)$ in $L_2$. Then $H_2^+$ and $H_2^-$ are invariant subspaces of the operators $B_1$ and $B_2$, respectively. Since $B_2$ is a finite Blaschke product, the subspace $H_2^- \ominus B_2^+H_2^-$ is finite-dimensional, and hence the operator $\pi_- - B_2^+\pi_- B_2$ which is the projection operator onto this subspace is finite-dimensional. Using the equation $\pi_+ + \pi_- = I$ and noting that $\pi_+ - B_1\pi_+ B_1^+$ is the projection operator onto the subspace $(H_2^+ \ominus B_1H_2^+)$ in $H_2^+$, we have, for the operator $\tilde{Q}$ defined by the formula (32),

$$\tilde{Q} = \pi_+ S_{\zeta\eta} \pi_- S_{\zeta\eta} \pi_+ = \pi_+ B_1 B_2^+ \pi_- B_2 B_1^+ \pi_+$$

$$= \pi_+ B_1[\pi_- - B_2^+ \pi_- B_2] B_1^+ \pi_+ = \pi_+ - B_1 \pi_+ B_1^+ = \pi_+ - B_1 \pi_+ B_1^+ \pi_- - B_2^+ \pi_- B_2 B_1^+ \pi_+.$$

The self-adjoint operator $\pi_+ B_1[\pi_- - B_2^+ \pi_- B_2] B_1^+ \pi_+$, mapping the subspace $H_2^+$ into itself nullifies a vector of $B_1 H_2^+ \ominus H_2^-$, since the operator $B_1^+ \pi_+ B_1$ is the projection operator onto the subspace $B_1^+ H_2^+ \ominus H_2^+$, and for any vector of the form $B_1 f$, where $f \in H_2^+$,

$$\pi_+ B_1[\pi_- - B_2^+ \pi_- B_2] B_1^+ \pi_+ (B_1 f) = \pi_+ B_1[\pi_- - B_2^+ \pi_- B_2] B_1^+ \pi_+ (B_1 f) = \pi_+ B_1[\pi_- - B_2^+ \pi_- B_2] f = 0.$$
Since $\pi_+ - B_1\pi_+ B_1^* - \hat{P}$ is the projection operator onto

$$H_2^0 \ominus \left( B_1 H_2^0 \oplus \bigoplus_{j=1}^{n} \{ e_j \} \right),$$

$$\tilde{Q}^k = [\pi_+ - B_1\pi_+ B_1^* - \hat{P}]^k + A^k = \pi_+ - B_1\pi_+ B_1^* - \hat{P} + A^k$$

and

$$\lim_{k \to \infty} \tilde{Q}^k = \pi_+ - B_1\pi_+ B_1^* - \hat{P} - \lim_{k \to \infty} \sum_{j=1}^{n} (1 - \lambda_j)^{k}(\cdot, e_j) e_j = \pi_+ - B_1\pi B_1^* - \hat{P}.$$

Therefore, if the function $S_{\xi\eta}(\lambda)$ can be factored in the form (33), then the solution of the problem under consideration reduces to finding the eigenvectors $\{ e_j \}$ of the finite-dimensional operator $\pi_+ B_1[\pi_+ - B_2^2 \pi_+ - B_2]B_2^2\pi_+$ and constructing the operator $\hat{P} = \sum_{j=1}^{n} (\cdot, e_j) e_j$. Then the desired best linear prediction $\xi(t)$ of the process $\xi(t)$, $t > 0$, is computed from the formula

$$\hat{\xi}(t) = \int_{-\infty}^{\infty} e^{ix\tau} \left\{ 1 - \frac{[(\pi^+ - B_1\pi_+ B_1^* - \hat{P})\tilde{g}(t, \cdot)](\lambda)}{\Gamma_{\xi}(\lambda)} \right\} dE_{\xi}(0),$$

where $\tilde{g}(\tau, \lambda) = e^{ix\tau}\Gamma_{\xi}(\lambda)$.

Note that in order to be able to factor the function $S_{\xi\eta}(\lambda)$ in the form (33), it is necessary and sufficient that the cross-spectral density $f_{\xi\eta}(\lambda)$ of the equivalent processes $\xi(t)$ and $\eta(t)$ can be factored in the form $f_{\xi\eta}(\lambda) = e^{ix\tau}h(\lambda)/B(\lambda)$, where $h(\lambda)$ is the boundary value of the function of class $H_1^0$ which is analytic in the lower half-plane, and $B(\lambda)$ is a finite Blaschke product which is analytic in the lower half-plane. We also point out that a representation in the form (33) of the function $S_{\xi\xi}(\lambda)$ arising in the solution of the linear interpolation problem for the process $\xi(t)$ is possible, if the density $f_{\xi\xi}(\lambda)$ is rational, and only in this case.

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