We are going to prove the following theorem:

**Theorem:**

Suppose \( p_0 > p_1 > \cdots > p_n > 0 \).

All zeros of the polynomial \( P(z) = \sum_{j=0}^{n} p_j z^j \) lie in \( \{ |z| > 1 \} \).

**Proof:** Let \( a_k = \frac{p_k - p_{k+1}}{p_0}, \ k = 0, \cdots, n \) and \( a_n = \frac{p_n}{p_0} \). Then we see that

\[
a_k > 0 \text{ and } \sum_{k=0}^{n} a_k = 1.
\]

\( a_0 = \frac{p_0 - p_1}{p_0} \) implies \( p_1 = 1 - a_0 \). \( a_1 = \frac{p_1 - p_2}{p_0} \) implies \( p_2 = 1 - a_0 - a_1 \).

In the same way we have

\[
\frac{p_k}{p_0} = 1 - (a_0 + a_1 + \cdots + a_{k-1}), \quad k = 1, \cdots, n.
\]

From this we have

\[
p_0^{-1} P(z) = \frac{p_0 + p_1 z + \cdots + p_n z^n}{p_0} = 1 + (1 - a_0) z + (1 - a_0 - a_1) z^2 + \cdots + (1 - a_0 - a_1 - \cdots a_{n-1}) z^n
\]

\[
= 1 + z + z^2 + \cdots + z^n - a_0 z (1 + z + \cdots + z^{n-1}) - a_1 z^2 (1 + z + \cdots + z^{n-2}) \cdots - a_{n-1} z^n.
\]

Multiply both sides by \( 1 - z \) we have

\[
p_0^{-1} P(z) (1 - z) = \frac{p_0 + p_1 z + \cdots + p_n z^n}{p_0} = 1 - z^{n+1} - a_0 z (1 - z^n) - a_1 z^2 (1 - z^{n-1}) - \cdots - a_{n-1} z^n (1 - z)
\]

\[
= 1 - z^{n+1} - z (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}) + z^{n+1} (a_0 + a_1 + \cdots + a_{n-1})
\]

\[
= 1 - z^{n+1} - z (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}) + z^{n+1} (1 - a_n), \text{ which means}
\]

\[
p_0^{-1} P(z) (1 - z) = 1 - z (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n).
\]
If $\alpha$ is a zero of $P(z)$, i.e. $P(\alpha) = 0$, then we have

$$1 = \alpha\left(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n\right).$$

Since $P(0) = P_0 > 0$, it must be $\alpha \neq 0$.

Hence

$$\frac{1}{|\alpha|} = \left|a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + a_n\alpha^n\right| \leq a_0 + a_1|\alpha| + \cdots + a_{n-1}|\alpha|^{n-1} + a_n|\alpha|^n.$$

If $|\alpha| < 1$, then from this inequality we have

$$a_0 + a_1|\alpha| + \cdots + a_{n-1}|\alpha|^{n-1} + a_n|\alpha|^n < a_0 + a_1 + \cdots + a_{n-1} + a_n = 1$$

we have a contradiction $\frac{1}{|\alpha|} < 1$. Thus $|\alpha| \geq 1$. To finish the proof, it remains to show $|\alpha| \neq 1$. On the contrary let's suppose $\alpha = e^{i\theta}$. Then we have

$$1 = a_0e^{i\theta} + a_1e^{2i\theta} + \cdots + a_{n-1}e^{(n-1)i\theta} + a_ne^{(n+1)i\theta}.$$  Comparing the real parts, we have

$$1 = a_0\cos\theta + a_1\cos2\theta + \cdots + a_{n-1}\cos(n-1)\theta + a_n\cos(n+1)\theta.$$  However we see that $\sum_{k=0}^n a_k = 1$, $\cos k\theta \leq 1$ and $\cos \theta < 1$ unless $\theta = 0(\text{mod } 2\pi)$. By inspection the equality $1 = a_0\cos\theta + a_1\cos2\theta + \cdots + a_{n-1}\cos(n-1)\theta + a_n\cos(n+1)\theta$ holds if and only if $\theta = 0(\text{mod } 2\pi)$ or $\alpha = 1$. It is clear $P(1) \neq 0$, so we conclude $|\alpha| > 1$. The proof is completed. $\blacksquare$