Kakeya-Enestöm's theorem

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We are going to prove the following theorem:

> Theorem:
> ***Suppose*** $p_0 > p_1 > \cdots > p_n > 0$.
> ***All zeros of the polynomial*** $P(z) = \sum_{j=0}^{n} p_j z^j$ ***lie in*** $\{|z| > 1\}$.

Proof) Let $a_k = \dfrac{p_k - p_{k+1}}{p_0}$, $k = 0, \cdots, n$ and $a_n = \dfrac{p_n}{p_0}$. Then we see that

$$a_k > 0 \text{ and } \sum_{k=0}^{n} a_k = 1 \ .$$

$a_0 = \dfrac{p_0 - p_1}{p_0}$ implies $\dfrac{p_1}{p_0} = 1 - a_0$. $a_1 = \dfrac{p_1 - p_2}{p_0}$ implies $\dfrac{p_2}{p_0} = 1 - a_0 - a_1$.

In the same way we have

$$\frac{p_k}{p_0} = 1 - (a_0 + a_1 + \cdots + a_{k-1}) \ , \ k = 1, \cdots, n \ .$$

From this we have

$$p_0^{-1} P(z) = \frac{p_0 + p_1 z + \cdots + p_n z^n}{p_0} = 1 + (1 - a_0) z + (1 - a_0 - a_1) z^2 + \cdots + (1 - a_0 - a_1 - \cdots a_{n-1}) z^n$$

$$= 1 + z + z^2 + \cdots + z^n - a_0 z (1 + z + \cdots + z^{n-1}) - a_1 z^2 (1 + z + \cdots + z^{n-2}) \ \cdots - a_{n-1} z^n \ .$$

Multiply both sides by $1 - z$ we have

$$p_0^{-1} P(z)(1 - z) = \frac{p_0 + p_1 z + \cdots + p_n z^n}{p_0} = 1 - z^{n+1} - a_0 z (1 - z^n) - a_1 z^2 (1 - z^{n-1}) - \cdots - a_{n-1} z^n (1 - z)$$

$$= 1 - z^{n+1} - z (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}) + z^{n+1} (a_0 + a_1 + \cdots + a_{n-1})$$

$= 1 - z^{n+1} - z (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}) + z^{n+1} (1 - a_n)$, which means

$$p_0^{-1} P(z)(1 - z) = 1 - z (a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n) .$$

If α is a zero of $P(z)$, i.e. $P(\alpha)=0$, then we have

$1=\alpha\left(a_0+a_1\alpha+\cdots+a_{n-1}\alpha^{n-1}+a_n\alpha^n\right)$. Since $P(0)=p_0>0$, it must be $\alpha\neq 0$.

Hence $\frac{1}{|\alpha|}=\left|a_0+a_1\alpha+\cdots+a_{n-1}\alpha^{n-1}+a_n\alpha^n\right|\leq a_0+a_1|\alpha|+\cdots+a_{n-1}|\alpha|^{n-1}+a_n|\alpha|^n$.

If $|\alpha|<1$, then from this inequality we have

$$a_0+a_1|\alpha|+\cdots+a_{n-1}|\alpha|^{n-1}+a_n|\alpha|^n<a_0+a_1+\cdots+a_{n-1}+a_n=1$$

we have a contradiction $\frac{1}{|\alpha|}<1$. Thus $|\alpha|\geq 1$. To finish the proof, it remains to show $|\alpha|\neq 1$. On the contrary let's suppose $\alpha=e^{i\theta}$. Then we have $1=a_0e^{i\theta}+a_1e^{2i\theta}+\cdots+a_{n-1}e^{in\theta}+a_ne^{i(n+1)\theta}$. Comparing the real parts, we have $1=a_0\cos\theta+a_1\cos 2\theta+\cdots+a_{n-1}\cos n\theta e^{in\theta}+a_n\cos(n+1)\theta$. However we see that $\sum_{k=0}^{n}a_k=1$, $\cos k\theta\leq 1$ and $\cos\theta<1$ unless $\theta=0(\operatorname{mod}2\pi)$. By inspection the equality $1=a_0\cos\theta+a_1\cos 2\theta+\cdots+a_{n-1}\cos n\theta e^{in\theta}+a_n\cos(n+1)\theta$ holds if and only if $\theta=0(\operatorname{mod}2\pi)$ or $\alpha=1$. It is clear $P(1)\neq 0$, so we conclude $|\alpha|>1$. The proof is completed.∎