A Consideration on Kerawala’s Method for Poncelet Porism in Two Circles

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1. We have two circles with radius \( R \) and \( r \), respectively: \( S \equiv x^2 + y^2 - R^2 = 0 \) and \( s \equiv (x - d)^2 + y^2 - r^2 = 0 \) in the x-y plane. Although Kerawala treated various cases in a general set up, we restrict here our study to the standard case where the large circle \( S \) completely contains the small circle \( s \) inside, that is we will assume \( R - r > |d| \). If the chord joining the points \( A_i(R \cos \theta_i, R \sin \theta_i) \) and \( A_{i+1}(R \cos \theta_{i+1}, R \sin \theta_{i+1}) \) on the circle \( S \) touches the circle \( s \), we must have

\[
R \cos \frac{1}{2}(\theta_{i+1} - \theta_i) - d \cos \frac{1}{2}(\theta_{i+1} + \theta_i) = r,
\]

which is easily found by drawing the picture. On writing \( t_i = \tan \frac{\theta_i}{4} \), we have the following lemma.

Lemma 1.1

\[
(1.2) \quad (R - d + r)(t_i + t_{i+1})^2 + (R + d + r)(1 - t_i t_{i+1})^2 = 2R(1 + t_i t_{i+1})^2
\]

\( i = 0, 1, 2 \cdots \)

Proof) Substituting

\[
\cos \left(\frac{\theta_{i+1} - \theta_i}{2}\right) = \frac{1 - \tan^2\frac{\theta_{i+1} - \theta_i}{4}}{1 + \tan^2\frac{\theta_{i+1} - \theta_i}{4}} = \frac{(1 + t_i t_{i+1})^2 - (t_{i+1} - t_i)^2}{(1 + t_i t_{i+1})^2 + (t_{i+1} - t_i)^2}
\]

\[
\cos \left(\frac{\theta_{i+1} + \theta_i}{2}\right) = \frac{1 - \tan^2\frac{\theta_{i+1} + \theta_i}{4}}{1 + \tan^2\frac{\theta_{i+1} + \theta_i}{4}} = \frac{(1 - t_i t_{i+1})^2 - (t_{i+1} + t_i)^2}{(1 - t_i t_{i+1})^2 + (t_{i+1} + t_i)^2}
\]

into (1.1), we have

\[
R \left\{ (1 + t_i t_{i+1})^2 - (t_{i+1} - t_i)^2 \right\} - d \left\{ (1 - t_i t_{i+1})^2 - (t_{i+1} + t_i)^2 \right\}
\]
\begin{align*}
&= r \left\{ (1 - t_i t_{i+1})^2 + (t_{i+1} + t_i)^2 \right\}, \\
&\text{from which we have the desired result.} \tag*{\phantom{\text{.}}}
\end{align*}

Now let \(2u^2 = R - d + r, \ 2v^2 = R + d + r, \ w^2 = R.\) Then we rewrite (1.2) as
\[(1.3) \quad u^2 (t_i + t_{i+1})^2 + v^2 (1 - t_i t_{i+1})^2 = w^2 (1 + t_i t_{i+1})^2\]
And similarly for \(t_i\) and \(t_{i-1}\) we have
\[u^2 (t_i + t_{i-1})^2 + v^2 (1 - t_i t_{i-1})^2 = w^2 (1 + t_i t_{i-1})^2.\]
No loss of generality if we set \(\theta_0 = 0\) or \(t_0 = 0\), then we have
\[(1.4) \quad t_i^2 u^2 = w^2 - v^2.\]
Hence considering the quadratic equation of the unknown \(x:\)
\[(1.5) \quad u^2 (t_i + x)^2 + v^2 (1 - t_i x)^2 = w^2 (1 + t_i x)^2,\]
we have two roots \(x = t_{i-1}\) and \(x = t_{i+1}.\) Since (1.5) becomes
\[(1.6) \quad \left( u^2 + (v^2 - w^2) t_i^2 \right) x^2 + 2t_i \left( u^2 - v^2 - w^2 \right) x + \left( u^2 t_i^2 + v^2 - w^2 \right), \]
\[= u^2 \left( 1 - t_i^2 t_{i-1}^2 \right) x^2 + 2t_i \left( u^2 - v^2 - w^2 \right) x + u^2 \left( t_i^2 - t_{i-1}^2 \right) = 0,\]
we have by using relation between coefficients and zeros of the above equation,
\[(1.7) \quad t_{i-1} + t_{i+1} = \frac{-u^2 + v^2 + w^2}{u^2} \frac{2t_i}{1 - t_i^2 t_{i-1}^2} \]
and
\[(1.8) \quad t_{i-1} t_{i+1} = \frac{t_i^2 - t_{i-1}^2}{1 - t_i^2 t_{i-1}^2} \]
so that
\[(1.9) \quad \frac{t_{i-1} + t_{i+1}}{1 - t_{i-1} t_{i+1}} = \frac{-u^2 + v^2 + w^2}{u^2 - v^2 + w^2} \frac{2t_i}{1 - t_i^2}.\]
We will notice here the relation between \(t_{i-1}\) and \(t_{i+1}\) is similar to the relation (1.3) between \(t_i\) and \(t_{i+1}.\) In fact, we can get the exactly same form as (1.3) except for the coefficients \(a^2, b^2, c^2\) instead of \(u^2, v^2, w^2.\)
Lemma 1.2

\[(1.10) \quad a^2(t_{i-1} + t_{i+1})^2 + b^2(1 - t_{i-1}t_{i+1})^2 = c^2(1 + t_{i-1}t_{i+1})^2,\]

where \(a^{-1} = R + d, \ b^{-1} = R - d, \ c^{-1} = r, \ i = 0, 1, 2 \ldots\)

By (1.4), \(t_i^2 = \frac{v^2 - w^2}{u^2}, \ 1 + t_i^2 = \frac{u^2 + w^2 - v^2}{u^2} = \frac{2(R - d)}{R - d + r}\) and

\[1 - t_i^2 = \frac{u^2 - w^2 + v^2}{u^2} = \frac{2r}{R - d + r}. \]

We have from (1.7)

\[t_{i-1} + t_{i+1} = \frac{-u^2 + v^2 + w^2}{u^2} \times \frac{2t_i}{1 - t_i^2t_i^2} = \frac{2(R + d)}{R - d + r} \times \frac{2t_i}{1 - t_i^2t_i^2}. \]

From (1.8)

\[1 - t_{i-1}t_{i+1} = 1 - \frac{t_i^2 - t_i^2}{1 - t_i^2t_i^2} = \frac{(1 - t_i^2)(1 + t_i^2)}{1 - t_i^2t_i^2} \quad \text{and} \quad 1 + t_{i-1}t_{i+1} = \frac{(1 + t_i^2)(1 - t_i^2)}{1 - t_i^2t_i^2}.

Hence

\[a^2(t_{i-1} + t_{i+1})^2 + b^2(1 - t_{i-1}t_{i+1})^2 - c^2(1 + t_{i-1}t_{i+1})^2\]

\[= \left(\frac{1}{R + d}\right)^2 \frac{4(R + d)^2 \times 4t_i^2}{(R - d + r)^2(1 - t_i^2t_i^2)^2} + \left(\frac{1}{R - d}\right)^2 \frac{(1 - t_i^2)^2}{(1 - t_i^2t_i^2)^2} \times \frac{4(R - d)^2}{(R - d + r)^2}

- \frac{1}{r^2} \frac{(1 + t_i^2)^2}{(1 - t_i^2t_i^2)^2} \times \frac{4t_i^2}{(R - d + r)^2} = 0\]

From (1.10) we have counterparts of (1.7)-(1.9):

\[(1.11) \quad t_{i-2} + t_{i+2} = \frac{-a^2 + b^2 + c^2}{a^2} \times \frac{2t_i}{1 - t_i^2t_i^2},\]

\[(1.12) \quad t_{i-2}t_{i+2} = \frac{t_i^2 - t_i^2}{1 - t_i^2t_i^2},\]

\[(1.13) \quad t_{i-2} + t_{i+2} = \frac{-a^2 + b^2 + c^2}{a^2 - b^2 + c^2} \times \frac{2t_i}{1 - t_i^2}.\]