Laplace’s equation in polar coordinates

Laplace’s equation in polar coordinates is written by

\[ r^2 \frac{\partial^2 U}{\partial r^2} + r \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} = 0 \]

where Laplace’s equation is

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \]

and

\[ u = u(x, y) = u(r \cos \theta, r \sin \theta) = U(r, \theta) = U \]

\[ x = r \cos \theta, \quad y = \sin \theta. \]

Notice here that confusions often happen if we usually use the same symbol \( u \) as \( U \).

Proof: We have easily

\[ r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \left( \frac{y}{x} \right). \]

From this

\[ \frac{\partial x}{\partial \theta} = \frac{x}{r} = \cos \theta, \quad \frac{\partial x}{\partial r} = \frac{y}{\sqrt{x^2 + y^2}} = \sin \theta, \]

\[ \frac{\partial y}{\partial \theta} = \frac{-x}{r} = \frac{\sin \theta}{r}, \quad \frac{\partial y}{\partial r} = \frac{y}{\sqrt{x^2 + y^2}} = \sin \theta, \]

\[ \frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2 + y^2}} \frac{x}{(x^2 + y^2)^{3/2}} = \frac{\sin^2 \theta}{r}, \quad \frac{\partial r}{\partial y} = \frac{1}{\sqrt{x^2 + y^2}} \frac{y}{(x^2 + y^2)^{3/2}} = \cos^2 \theta, \]

\[ \frac{\partial \theta}{\partial x} = \frac{-2xy}{r^2}, \quad \frac{\partial \theta}{\partial y} = \frac{-2xy}{r^2}. \]

Now we carry out the following calculations:

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial x} \frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial x} \right) = \frac{\partial^2 U}{\partial r^2} \left( \frac{\partial r}{\partial x} \right)^2 + \frac{\partial U}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 U}{\partial \theta^2} \left( \frac{\partial \theta}{\partial x} \right)^2 + \frac{\partial U}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2}. \]

\[ \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial x} \frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta} \frac{\partial \theta}{\partial y} \right) = \frac{\partial^2 U}{\partial r^2} \left( \frac{\partial r}{\partial y} \right)^2 + \frac{\partial U}{\partial r} \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 U}{\partial \theta^2} \left( \frac{\partial \theta}{\partial y} \right)^2 + \frac{\partial U}{\partial \theta} \frac{\partial^2 \theta}{\partial y^2}. \]

Then we have

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 U}{\partial r^2} \left( \frac{1}{r} \right)^2 + \frac{1}{r} \frac{\partial U}{\partial r} \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} = 0, \]

from which we have the conclusion.\( \blacksquare \)