ラプラシアン$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$　は極座標で

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

となる。

証明）$x = \cos\theta$　，$y = \sin\theta$　より

$$r = \sqrt{x^2 + y^2} \quad , \theta = \arctan\frac{y}{x}$$

したがって、

$$\frac{\partial r}{\partial x} = \cos\theta \quad , \quad \frac{\partial r}{\partial y} = \sin\theta \, , \quad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad , \quad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \quad ,$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)$$

$$= \cos\theta\left(\cos\theta\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2}{\partial \theta^2}\right)$$

$$-\frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{\partial^2}{\partial r\partial \theta} - \frac{\cos\theta}{r}\frac{\partial}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2}{\partial \theta^2}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + 2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta} - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2}$$

同様に、

$$\frac{\partial^2}{\partial y^2}=\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$=\sin\theta\left(\sin\theta\frac{\partial^2}{\partial r^2}-\frac{\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2}{\partial\theta^2}\right)$$

$$+\frac{\cos\theta}{r}\left(\cos\theta\frac{\partial}{\partial r}+\sin\theta\frac{\partial^2}{\partial r\partial\theta}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r}\frac{\partial^2}{\partial\theta}\right)$$

$$=\sin^2\theta\frac{\partial^2}{\partial r^2}-2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial r}+2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

もういちど並べて書くと、

$$\frac{\partial^2}{\partial x^2}=\cos^2\theta\frac{\partial^2}{\partial r^2}+2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}-2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\frac{\partial^2}{\partial y^2}=\sin^2\theta\frac{\partial^2}{\partial r^2}-2\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}$$

この２つを加えると

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}$$

となり望む結果が得られた。