lesson 9

last time:

$F$: field (e.g. $\mathbb{R}$, $\mathbb{Z}/p\mathbb{Z}$)

$V$: vector space over $F$:

$T: V \rightarrow W$ homomorphism of $F$-vector spaces

§ Spans, Linear independence, bases

$(v_1, v_2, \ldots, v_n)$ ordered finite set of vectors in $V$,

$S = \{v_1, v_2, \ldots, v_n\}$ set of vectors (Forget the order)

Linear combination of ordered set: $w = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$, $a_i \in F$

all such $w$ define $W = \text{span of } S$.

Fact this is a subspace of $V$.

( since $w + w' = (a_i + b_i)v_i + \cdots + (a_n + b_n)v_n$, $cw = (ca_i)v_i + \cdots + (ca_n)v_n$ )

Convention If $S = \emptyset$, the span of $S = \{0\} \subset V$. (will come up in the inductive arguments).

Definition $V$ is finite dimensional if there is a finite set $S$ of vectors in $V$ with span of $S = V$.

Example: $V = F^n$ is finite dimensional

$v_1 = (1, 0, \ldots, 0)$

$v_2 = (0, 1, \ldots, 0)$

\vdots

$v_n = (0, 0, \ldots, 1)$

$(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i v_i$
Non-example

\( V = F[X] \) is no way finite dimensional (use degree of polynomials)

**Linear independence**

\( \{v_1, v_2, \ldots, v_n\} \) is linearly independent if the relation

\[ a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \]

only holds when \( a_1 = a_2 = \cdots = a_n = 0 \).

**Example:** \( V = \mathbb{R}^3 : v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (1,2,3) \)

\( \text{Span}\{v_1, v_2\} = \{(a,b,0) : a,b \in \mathbb{R}\} \), in fact \( (a,b,0) = (a-b)v_1 + bv_2 \).

On the other hand, \( (v_1, v_2, v_3) \) are linearly independent

because if \( a_1v_1 + a_2v_2 + a_3v_3 = 0 \), then \( 3a_3 = 0 \) or \( a_3 = 0 \),

likewise \( a_2 = 0 \) and \( a_1 = 0 \).

**Definition:** We say an ordered set \( (v_1, v_2, \ldots, v_n) \) is a basis of \( V \)

if it spans \( V \) and is linearly independent.

What this means: every vector \( w \in V \) is uniquely expressed as a linear combination

\( w = a_1v_1 + a_2v_2 + \cdots + a_nv_n \)

( because suppose also

\( w = b_1v_1 + b_2v_2 + \cdots + b_nv_n \),

then

\[ 0 = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n \]

so

\[ 0 = (a_1 - b_1)(a_2 - b_2) = \cdots = (a_n - b_n) \).
Let \( w, w' \in V \).

A basis gives rise to an isomorphism of vector spaces

\[
V \longrightarrow f^* \quad f
\]

\[
f(w) = (a_1, \ldots, a_n).
\]

where \( w = a_1 v_1 + \cdots + a_n v_n \)

is unique expression in

given basis.

If \( f(w') = (b_1, \ldots, b_n) \). \( f(w + w') = (a_i + b_i, \ldots, a_n + b_n) = f(w) + f(w') \) and

\[
f(cw) = (ca_1, \ldots, ca_n) = cf(w)
\]

which means it is a homomorphism of vector

spaces. Moreover

\[
\text{Onto} \iff \text{span} \quad \text{1-to-1} \iff \text{lin. indep.}
\]

hence \( f \) gives us an isomorphism .

People can't see their own future. But you can see other people's future. It is an interesting time question. How do you know other people's future. For example, if you drive on the high way., on the other side road you see the five miles backup of your trip will be the other side's future. You can't see your future. I know your future in this course.
**Theorem**

If \( S = \{v_1, v_2, \cdots, v_n\} \) is a finite set which spans \( V \), then a subset of \( S \) gives a basis for \( V \).

**Proof** If the elements of \( S \) are linearly independent, then we’re done. If not, we have a relation \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) with some \( a_i \neq 0 \). We can reorder so that \( a_n \neq 0 \) and thus \( a_n^{-1} \) exists. Then

\[
a_n v_n = -(a_1v_1 + a_2v_2 + \cdots + a_{n-1}v_{n-1})
\]

in \( V \). Multiply by \( a_n^{-1} \), then we have

\[
v_n = \left( -\frac{a_1}{a_n} v_1 \right) + \left( -\frac{a_2}{a_n} v_2 \right) + \cdots + \left( -\frac{a_{n-1}}{a_n} v_{n-1} \right).
\]

Hence \( V = \text{span} S = \text{span}\{v_1, v_2, \cdots, v_{n-1}\} \) If this new set \( \{v_1, v_2, \cdots, v_{n-1}\} \) is linearly independent, we’re done. If not, we repeat until we’re done. We must finish in a finite number of steps because \( S \) was finite to start with. □

**Theorem**

If \( L \) is linearly independent set of vectors, it can be extended to form a basis of \( V \).

**Proof** If \( L \) spans \( V \), then done. If not, let \( S \) be a finite set spanning \( V \). There must be some \( v \in S \) such that \( v \notin \text{Span} L \) (since otherwise \( \text{Span} L = \text{Span} S = V \)). Then we claim \( L' = L \cup \{v\} \) is linear independent.

Why? Suppose \( L = \{w_1, w_2, \cdots, w_n\} \) and \( \sum a_iw_i + bv = 0 \). Then either \( b = 0 \) or \( v = -\frac{1}{b} \sum a_iw_i \in \text{Span} L \) (contradiction). Hence \( b = 0 \) and \( \sum a_iw_i = 0 \Rightarrow a_i = 0 \) since \( L \) was linearly independent set of vectors. If \( L' = L \cup \{v\} \) spans, we are done. Otherwise we keep adjoining vectors from the finite set \( S \) until done. □
Main Theorem

If \( S = \{v_1, v_2, \ldots, v_n\} \) spans \( V \), \( L = \{w_1, w_2, \ldots, w_m\} \) is linear independent in \( V \),
then \( n \geq m \).

Proof)

Since \( S \) spans, we may write each element \( w_j \) of \( L \) as

\[
 w_j = \sum_{i=1}^{n} a_{ij} v_i .
\]

Try to make a non-trivial linear relation on \( w_j \):

\[
 0_v = \sum_{j=1}^{m} c_j w_j = \sum_{j=1}^{m} c_j a_{ij} v_i = \sum_{i=1}^{n} \left( \sum_{j=1}^{m} c_j a_{ij} \right) v_i .
\]

If \( c_j = 0 \) for all \( j \), we are done.

If not, we can arrange that \( \sum_{j=1}^{m} a_{ij} c_j = 0 \) with some \( c_j \neq 0 \), and the \( w_j \)
could not be linear independent. We are dealing with a system of \( n \) linear equations and \( m \) unknowns. If \( m > n \) (more unknowns \( c_j \) than the equations index by \( i \)) we can find a nontrivial solution. For example consider two equation \( 2x + 3y + 4z = 0 \) and \( 2x + 7y + 8z = 0 \). Eliminate \( x \), we have \( 4y + 4z = 0 \) and we have nontrivial solution \( y = 1, z = -1 \). See linear algebra book. This is a contradiction to our hypothesis which means \( n \geq m \).

Corollary

1) All bases of \( V \) have the same number of elements \( =: \dim (V) \)
2) All spanning sets \( S \) have \( \#S \geq \dim V \)
3) All linearly independent sets \( L \) have \( \#L \leq \dim V \)

Proof of Corollary)

1) Two basis \( B \) and \( B' \). Since \( B \) spans \( V \) and \( B' \) is linearly independent, \( \#B \geq \#B' \). Since \( B' \) spans \( V \) and \( B \) is linearly independent, \( \#B' \geq \#B \).

So \( \#B = \#B' \)
2) and 3) are immediate.

Remark: \( \dim V \geq 0 \), \( \dim \{0\} = 0 \) and \( \dim (F^n) = n \). Use a basis \( v_1 = (1, 0, \ldots, 0) \)

\( v_2 = (0, 1, \ldots, 0) \), \( v_n = (0, 0, \ldots, 1) \).

Basis is a key analyzing the vector space.
Propositon

Suppose \( W \subseteq V \) is finite dimensional and \( \{w_1, \ldots, w_m\} \) is a bases for \( W \).

Then we may extend this to a basis \( \{w_1, \ldots, w_m, v_{m+1}, v_{m+2}, \ldots, v_n\} \) for \( V \).

Proof) The basis is linear independent in \( V \) so can be extended to a basis.

\( W \subseteq V \) gives \( f : V \rightarrow V/W \) quotient vector space. Then \( \{w_1, \ldots, w_m\} \) are in kernel of \( f \) or \( f(w_1) = f(w_2) = \cdots = f(w_m) = 0 \).

Fact \( (f(v_{m+1}), f(v_{m+2}), \ldots, f(v_n)) \) gives a basis for \( V/W \).

\( W' = \text{Span of } \{v_{m+1}, \ldots, v_n\} \) is subspace of \( V \) mapping isomorphically to \( V/W \).

Hence we have \( \dim V = \dim W + \dim (V/W) \).

Warning This is not true for general groups.

\[ \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \]

\[ \mathbb{Z}/4\mathbb{Z} \] is a cyclic group of order 4. \( \mathbb{Z}/4\mathbb{Z} \) is cyclic group of order 2. So quotient group is cyclic order. However only subgroup of order 2 is \( H = \mathbb{Z}/2\mathbb{Z} \).

See you Monday.