Abstract Algebra
Lecture 4 by Gross
§ Review
Many homomorphisms today.
Homomorphism is in some sense a most important concept in the group theory.

\[ f(x \cdot y) = f(x) \cdot f(y) \]

- \( G \rightarrow G' \) homomorphism \( \iff \)
  \[ f(e) = e', \ e \in G, \ e' \in G' \ (identity) \]

\[ ee = e, \ \text{so} \ f(e) \cdot f(e) = f(ee) = f(e) \Rightarrow f(e) = e' \]

\[ f(g^{-1}) = (f(g))^{-1} \ (inverse) \]

\[ gg^{-1} = e, \ \text{so} \ f(g) \cdot f(g^{-1}) = f(gg^{-1}) = f(e) = e' \]

You’ll need these soon.
- composition \( G \rightarrow G' \rightarrow G'' \)
  If \( h \) and \( f \) are homomorphisms., then composition of two homomorphisms \( h \circ f \) is the third homomorphism.
- Image of \( f = \{g' = f(g) | g \in G\} \subset G' \)
- Kernel of \( f = \{g \ | \ f(g) = e' \} \subset G, \ e' \in G' \ (identity) \)
- When Image of \( f = G' \) and Kernel of \( f = \{e\} \), we say \( f \) is an isomorphism.
- When \( G = G' \) and \( f \) is an isomorphism, we say \( f \) is an automorphism.
• Kernel and Image are subgroups but kernel is also a special kind of subgroup: it’s a **normal** subgroup $H$, denoted by $H \triangleleft G$ which means a nice property that $gHg^{-1} = H$ for all $g \in G$. $gHg^{-1}$ is a subgroup because multiplication of two elements in $gHg^{-1}$ is closed in $gHg^{-1}$:

$$(gh^{-1})(gh')^{-1} = ghg^{-1} \in gHg^{-1}.$$ Normal subgroup satisfies $gHg^{-1} = H$ for any $g \in G$. Hence another way of saying this: $H$ is "closed under conjugation for any $g \in G"$ where $ghg^{-1}$ is the conjugation of $h$ by $g$.

First this was noted by Galois.

• The kernels are normal:

Let $h \in \text{Kernel}(f)$ and $g \in G$. Then

$$f(g \cdot h \cdot g^{-1}) = f(g) \cdot f(h) \cdot f(g^{-1}) = f(g) \cdot e^h(f(g))^{-1} = e^h$$

hence $g \cdot h \cdot g^{-1} \in \text{Kernel}(f)$ for any $g \in G$, which means kernel is close under conjugation and is normal.

• In Abelian group, subspaces are always normal. But not all subgroups are normal!

**Example:** Let $G = S_3$ and consider a subspace $H = \{e, \tau\}$,

\[
\tau : \begin{cases} 
1 \to 1 \\
2 \to 2 \\
3 \to 3 
\end{cases}
\]

Making conjugation by

\[
\tau' : \begin{cases} 
1 \to 1 \\
2 \to 2 \\
3 \to 3 
\end{cases}
\]

gives us $\tau' \tau (\tau')^{-1} = \tau$.
More easily to see that $H = \{e, \tau\}$ is not normal, we evaluate

$\tau'\tau(\tau')^{-1}(3) = \tau'\tau(2) = \tau'(1) = 1$, $\tau'\tau(\tau')^{-1} \neq \tau$. So $H = \{e, \tau\}$ can't be the kernel of any homomorphism.

- Eventually we will see (big theorem): if $H \triangleleft G$ normal subgroup, then there is a homomorphism $f : G \to G'$ such that $\text{Kernel}(f) = H$.

§ Examples

1. $G = GL_n(\mathbb{R}) \to G' = \mathbb{R}^\ast = GL_1(\mathbb{R})$, $f$ homomorphism: $f(A) = \det A$.

$$\det(AB) = \det(A) \cdot \det(B)$$

$$\text{Image}(f) = \mathbb{R}^\ast$$

$$\det \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \lambda \in \mathbb{R}^\ast$$

$\text{Kernel}(f) = \{ A \in GL_n(\mathbb{R}) | \det A = 1 \}$ special linear group and $SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$ because a kernel is always a normal subgroup.

**Note:** Set of matrices with any fixed determinant value is stable under conjugation.

Let $\det A = a$, we have $\det(BAB^{-1}) = \det B \det A (\det B)^{-1} = \det A = a$.

2. Extremely important homomorphism.

$f : S_n \to GL_n(\mathbb{R})$

$f(\sigma) = A_\sigma$, which is called “permutation matrix associated to $\sigma$.”
Example: \( G = S_3 \),

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow
\end{array}
\begin{array}{c}
1 \\
2 \\
3
\end{array}
\Rightarrow \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

Book verifies: \( f(\sigma \tau) = A_\sigma A_\tau \)

Image \( f \) ="set of permutation matrices"

\[ \text{Ker}(f) = \{e\} \]

Fact \( \det(f(\sigma)) = \pm 1 \) for all \( \sigma \in S_n \) and both values occur.

\[ \det(f(e)) = \det(I) = 1 \quad \text{and} \quad \det(f(\tau)) = -1 \]

where

\[
\begin{array}{c}
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\end{array}
\]
③ Composition of ① and ② looks like this:

\[ S_n \rightarrow GL_n(\mathbb{R}) \rightarrow \mathbb{R}^\times \]

Image = \{±1\} ⊂ \mathbb{R}^\times

Kernel = \{σ: det(f(σ)) = ±1\} ⊂ S_n called “Alternating group” \( A_n \) will show \(|A_n| = |S_n|/2 = \frac{n!}{2} \) (which will be seen later)

\[ S_n \rightarrow GL_n(\mathbb{R}) \rightarrow \mathbb{R}^\times \] is “sign map” (Note: to avoid circularity, det is defined independently of formula involving signs of permutations)
The elements of \( A_n \) are called “even permutations”.

Note: There is also an alternative definition of this “sign map” (write permutation as product of transpositions; if even number of them \( \mapsto +1 \), if odd number of them \( \mapsto -1 \)

**Example:** In \( S_3 \) order 6

\[ \sigma \rightarrow \sigma^2 \] even terms
\[ \text{no } A_3 \text{-has order 3} \]

\[ \tau \rightarrow \tau'' \] odd permutation
§ Centers and Inner Automorphisms

Let $G$ a group.

$Z(G) =$ "center of $G" = \{ z \in G : zg = gz \ \forall g \in G \}$ whose elements commute with everything in $G$. This is a normal subgroups.

Example:

- $G=Z(G) \iff G$ is abelian
- $G=S_n \Rightarrow Z(G) = \{e\}$
- $G=GL_n(\mathbb{R}) \Rightarrow Z(G) = \{ \lambda I : \lambda \in \mathbb{R}^* \}$

Another homomorphism:

$G \rightarrow Aut(G) = \{ \text{all isomorphisms } h : G \rightarrow G \}$

Defined by $f(g)(h) = ghg^{-1} = \text{ conjugation by } g$

We can easily verify $f(g) \in Aut(G)$.

Actually

- $f(g)(hh') = ghh'g^{-1} = ghg^{-1} \cdot gh'g^{-1} = f(g)(h) \cdot f(g)(h')$
- Also $f$ is a homomorphism: $f(gg') = f(g) f(g')$ because $f(gg')(h) = (gg')h(gg')^{-1} = gg'hg^{-1}g^{-1} = (f(g)f(g'))(h)$.
- $f(g)$ is a bijection because it can exhibit inverse

\[
    f(g) \cdot f(g^{-1})(h) = f(g)(g^{-1}hg) = g(g^{-1}hg)g^{-1} = h
\]

\[
    f(g) \cdot f(g^{-1}) = id \quad \text{and} \quad f(g^{-1}) \cdot f(g) = id
\]
What’s the kernel of $f$. Center $Z(G)$, exactly.

$$\text{Kernel}(f) = \{g \in G : ghg^{-1} = f(g)h = h, \forall h \in G\} = Z(G)$$

Is $f$ surjective?

**Example:** $G = \text{Klein 4-group} = \{e, \tau_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tau_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\}$.  

Since $G$ is Abelian, $Z(G) = G$, so $\text{Image}(f)$ in $\text{Aut}(G)$ is $\{e\}$.  

However $\text{Aut}(G)$ is nontrivial! In fact

$$\text{Aut}(G) \cong S_3.$$  

If $g : G \to G$ an automorphism can associate to $g$ a permutation of $\{\tau_1, \tau_2, \tau_3\}$ This gives a homomorphism $\text{Aut}(G) \to S_3$ with trivial kernel.  

This map $\text{Aut}(G) \to S_3$ is also surjective (full image).

In general:

$$\text{Image} \left( f : G \to \text{Aut}(G) \right) \text{ is called “group of inner automorphisms” denoted by } \text{Inn}(G).$$  

Or we can write this

$$\text{Inn}(G) = \{a(h) = ghg^{-1} \text{ for some } g \in G\}.$$  

Try it this language.

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A.A.