Maximum of Exponential Distributed Random Variables

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1. Suppose that $X_1, X_2, \cdots X_n$ are mutually independent and identically distributed random variables with the standard exponential distribution

$$P(X_k \le x) = \begin{cases} 1 - e^{-x}, & x \ge 0 \\ 0 \quad , & x < 0 \end{cases} \quad . \quad k = 1, 2, \cdots \quad .$$

Define $M_n = \max\{X_1, \cdots, X_n\}$, and $S_n = X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n}$. $n = 1, 2, \cdots$

We can show that $M_n \overset{d}{=} S_n$, i.e. $P(M_n \le x) = P(S_n \le x)$ for every x .

Actually we are going to prove that M_n and S_n have the same characteristic function. We easily see that the distribution function of M_n is

$F(x) = P(M_n \le x) = (1 - e^{-x})^n$, and its density function

$f(x) = \frac{dF(x)}{dx} = n(1 - e^{-x})^{n-1} e^{-x}$. Hence the characteristic function of the distribution for M_n is by definition

$\varphi(t) = \int_{-\infty}^{\infty} f(x) e^{itx} dx = n\int_0^{\infty} (1 - e^{-x})^{n-1} e^{-x} e^{itx} dx = n\int_0^{\infty} (1 - e^{-x})^{n-1} e^{x(it-1)} dx$, and by a

partial integration , we get $\varphi(t) = \frac{n(n-1)}{1 - it} \int_0^{\infty} (1 - e^{-x})^{n-2} e^{x(it-2)} dx$ and by

repeating partial integrations we have finally $\varphi(t) = \frac{n!}{(1-it)(2-it)\cdots(n-it)}$.

On the other hand the distribution function of $\frac{X_k}{k}$ is $P\left(\frac{X_k}{k} \le x\right) = 1 - e^{-kx}$

and the characteristic function is $\varphi_k(t) = \int_0^{\infty} ke^{-kx} e^{itx} dx = \frac{k}{k - it}$. Since we have

already assumed $X_1, \frac{X_2}{2}, \frac{X_3}{3} \cdots$ are mutually independent , the characterisitic function of the sum of independent random variables

$S_n = X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n}$ is given by the product

$\varphi_1(t)\varphi_2(t)\cdots\varphi_n(t) = \frac{n!}{(1-it)(2-it)\cdots(n-it)}$ which is exactly $\varphi(t)$.■

2.

From the fact which is proved insection 1, the characteristic function of

$S_n - \log n$ is $Ee^{it(S_n - \log n)} = n^{it} Ee^{itS_n} = \frac{n^{it} n!}{(1-it)(2-it)\cdots(n-it)}$. It is known that

$\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\cdots(z+n)}$, where $\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx$. Hence we can see that

$\lim_{n\to\infty} Ee^{it(S_n - \log n)} = \Gamma(1-it)$ .On the other hand $\Gamma(1-it)$ is the characteristic

function of the Gumbell distribution $G(x) = \exp(-\exp(-x)), x \in \mathbb{R}$. Actually

$\varphi(t) = Ee^{itX} = \int_{-\infty}^{\infty} e^{itu} dG(u) = \int_{-\infty}^{\infty} e^{itu} e^{-u} e^{-e^{-u}} du = \int_0^\infty x^{-it} e^{-x} dx = \Gamma(1-it)$,where we

used the change of variables in the last integration. Thus we have the following theorem.

Theorem Suppose that $X_1, X_2, \cdots X_n$ are mutually independent and identically distributed random variables with the standard exponential distribution. Let $M_n = \max\{X_1, \cdots, X_n\}$. Then the distribution of $M_n - \log n$

approaches to the Gumbell distribution $G(x) = \exp(-\exp(-x)), x \in \mathbb{R}$ as

$n \to \infty$.

proof) The distribution of $M_n - \log n$ is same as $S_n - \log n$,where

$S_n = X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n}$ and we have shown above that the characteristic function of $S_n - \log n$ approaches to the characteristic function of the Gumbell distribution.■