Maximum of Exponential Distributed Random Variables
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1. Suppose that $X_1, X_2, \cdots, X_n$ are mutually independent and identically distributed random variables with the standard exponential distribution

$$P(X_k \leq x) = \begin{cases} 1 - e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad k = 1, 2, \cdots.$$ 

Define $M_n = \max\{X_1, \cdots, X_n\}$, and $S_n = X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n}, n = 1, 2, \cdots$

We can show that $M_n \overset{d}{=} S_n$, i.e. $P(M_n \leq x) = P(S_n \leq x)$ for every $x$.

Actually we are going to prove that $M_n$ and $S_n$ have the same characteristic function. We easily see that the distribution function of $M_n$ is

$$F(x) = P(M_n \leq x) = (1-e^{-x})^n,$$

and its density function

$$f(x) = \frac{dF(x)}{dx} = n(1-e^{-x})^{n-1}e^{-x}.$$ 

Hence the characteristic function of the distribution for $M_n$ is by definition

$$\varphi(t) = \int_{-\infty}^{\infty} f(x)e^{itx} dx = n \int_{0}^{\infty} (1-e^{-x})^{n-1}e^{-x}e^{itx} dx = n \int_{0}^{\infty} (1-e^{-x})^{n-1}e^{it(n-1)x} dx,$$

and by a partial integration, we get

$$\varphi(t) = \frac{n(n-1)}{1-it} \int_{0}^{\infty} (1-e^{-x})^{n-2} e^{it(n-2)x} dx$$

and by repeating partial integrations we have finally

$$\varphi(t) = \frac{n!}{(1-it)(2-it)\cdots(n-it)}.$$ 

On the other hand the distribution function of $\frac{X_k}{k}$ is $P\left(\frac{X_k}{k} \leq x\right) = 1 - e^{-kx}$ and the characteristic function is $\varphi_k(t) = \int_{0}^{\infty} ke^{-kx}e^{itx} dx = \frac{k}{k-it}$. Since we have already assumed $X_1, X_2, X_3, \cdots$ are mutually independent, the characteristic function of the sum of independent random variables
\[ S_n = X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n} \] is given by the product
\[ \varphi_1(t)\varphi_2(t)\cdots\varphi_n(t) = \frac{n!}{(1-it)(2-it)\cdots(n-it)} \] which is exactly \( \varphi(t) \).

2.

From the fact which is proved in section 1, the characteristic function of
\[ S_n - \log n \] is
\[ Ee^{it(S_n - \log n)} = n^n Ee^{itS_n} = \frac{n^n n!}{(1-it)(2-it)\cdots(n-it)} . \]

It is known that
\[ \Gamma(z) = \lim_{n \to \infty} \frac{n^n n!}{z(z+1)\cdots(z+n)} \]
where \( \Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx \). Hence we can see that
\[ \lim_{n \to \infty} Ee^{it(S_n - \log n)} = \Gamma(1-it) . \]

On the other hand \( \Gamma(1-it) \) is the characteristic function of the Gumbell distribution \( G(x) = \exp(-\exp(-x)), x \in \mathbb{R} \). Actually
\[ \varphi(t) = Ee^{itX} = \int_{-\infty}^\infty e^{itu}dG(u) = \int_{-\infty}^\infty e^{it\mu}e^{-u}e^{-e^{-u}}du = \int_0^\infty x^{-it}e^{-x}dx = \Gamma(1-it) \],
where we used the change of variables in the last integration. Thus we have the following theorem.

**Theorem** Suppose that \( X_1,X_2,\cdots,X_n \) are mutually independent and identically distributed random variables with the standard exponential distribution. Let \( M_n = \max\{X_1,\cdots,X_n\} \). Then the distribution of \( M_n - \log n \) approaches to the Gumbell distribution \( G(x) = \exp(-\exp(-x)), x \in \mathbb{R} \) as \( n \to \infty \).

proof) The distribution of \( M_n - \log n \) is same as \( S_n - \log n \), where
\[ S_n = X_1 + \frac{X_2}{2} + \frac{X_3}{3} + \cdots + \frac{X_n}{n} \] and we have shown above that the characteristic function of \( S_n - \log n \) approaches to the characteristic function of the Gumbell distribution.