Non-Analytic Functions of a Complex Variables

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1. Historical background

A function

\[ w = f(z) = \phi(x, y) + iv(x, y) \]

where \( w = u + iv \) and \( z = x + iy \), is said to be defined for a given region of \( z \) if \( w \) is determined whenever \( z \) is assigned a value in that region. The equation (1) is equivalent to the two real simultaneous equations

\[ \begin{align*}
\phi(x, y) = u(x, y), \\
v(x, y) = \psi(x, y),
\end{align*} \]

which themselves express a transformation of the \( xy \) plane onto \( uv \) plane.

We assume that \( \phi \) and \( \psi \) are continuous and they possess continuous first derivatives.

If \( z \) and \( w \) have corresponding increments \( \Delta z \) and \( \Delta w \), the limit of their ratio,

\[ \frac{\Delta w}{\Delta z}, \]

is the derivative \( \frac{dw}{dz} \) whenever it has a unique value for all methods of approach of \( \Delta z \) approaches zero, and we call it the directional derivative.

\[ \gamma = \alpha + i\beta = \frac{dw}{dz} = \lim_{\Delta z \to 0} \Delta w = \lim_{\Delta z \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} \]

\[ = \lim_{\Delta x \Delta y \to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y} = \frac{u_x + iv_y + m(u_x + iv_y)}{1 + im} \]

where \( m = \frac{dy}{dx} \) is the slope of the curve of approach.

Actually,

\[ \lim_{\Delta y \to 0} \frac{\Delta u}{\Delta x} = u_x + mu_y, \quad \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v_x + v_y \frac{dy}{dx} = v_x + mv_y. \]

**Proposition 1** In order that (3) gives a unique result independent of \( m \) is the necessary and sufficient condition are the well known Cauchy-Riemann equation

\[ u_x = v_y, \quad u_y = -v_x \]
Proof: In (3) if we put \( m = 0 \), we would have \( \gamma = u_x + iv_y \), and if we put \( m \to \infty \), we would have \( \gamma = -iu_y + v_x \). Hence we have \( u_x = v_y \) and \( u_y = -v_x \). Conversely assume Cauchy-Riemann equation, then we have

\[
\gamma \equiv \frac{u_x + iv_x + m(u_y + iv_y)}{1 + im} = \frac{u_x + iv_x + m(-v_y + iu_x)}{1 + im} = u_x + iv_x,
\]

which is seen to be independent of \( m \). The equation (3) may be written in other forms. Let \( \theta \) denote the angle between the x axis and the curve of approach, so that \( \tan \theta = m \); then, by means of the usual relation \( e^{i\theta} = \cos \theta + i \sin \theta \), it is reducing (3) to the form

Proposition 2.

(5) \[ \gamma = \frac{d \omega}{d \zeta} = \lim_{\Delta \to 0} \Delta w = D[f(z)] + Pf[f(z)] e^{-2i\theta} \]

where

(6) \[ D[f(z)] = \frac{1}{2} \left[ u_x + v_y + i(v_x - u_y) \right], \]

\[ Pf[f(z)] = \frac{1}{2} \left[ u_x - v_y + i(v_x + u_y) \right]. \]

Proof: Using the relations

\[
m = \tan \theta = \frac{\sin \theta}{\cos \theta} = -i \frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}} = -i \frac{1 - e^{2i\theta}}{1 + e^{2i\theta}},
\]

\[
1 + im = \frac{2}{1 + e^{-2i\theta}} \quad \text{and} \quad \frac{m}{1 + im} = \frac{1 - e^{-2i\theta}}{2i}
\]

we have

\[
\gamma = \frac{u_x + iv_x + m(u_y + iv_y)}{1 + im} = \frac{(1 + e^{-2i\theta})(u_x + iv_x) + (1 - e^{2i\theta})(u_y + iv_y)}{2i},
\]

which is reduced to the desired result (6).

Again we would have Cauchy-Riemann equation if \( \gamma \) was independent of the approach direction \( \theta \), that is, if we assumed that \( Pf[f(z)] = \frac{1}{2} \left[ u_x - v_y + i(v_x + u_y) \right] = 0 \).

2. Principal directions. Characteristic Lines

If we introduce the familiar notation

(7) \[ E = u_x^2 + v_x^2, \quad G = u_y^2 + v_y^2, \quad F = u_x u_y + v_x v_y, \quad J = u_x v_y - u_y v_x. \]

Then we have easily

...
(8) \( J^2 = EG - F^2 \).

And it follows readily that

\[
(9) \quad r = \frac{\left( \frac{dw}{dz} \right)^2}{\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}} = \frac{du^2 + dv^2}{dx^2 + dy^2} = \frac{\left( \frac{du}{dx} \right)^2 + \left( \frac{dv}{dx} \right)^2}{1 + \left( \frac{dy}{dx} \right)^2} = \frac{\left( u_x + \frac{u_y}{x} \right)^2 + \left( v_x + \frac{v_y}{x} \right)^2}{1 + m^2} = \frac{E + 2mF + Gm^2}{1 + m^2}
\]

Proposition 3. The maximum and minimum values of \( r \) are found to exist for values of \( m \) given by the equation

\[
(10) \quad F + (G - E)m - Fm^2 = 0,
\]

and these values of \( r \) are themselves solutions of the quadratic equation

\[
(11) \quad \rho^2 - (E + G) \rho + J^2 = 0
\]

Proof: Differentiating \( r = \frac{E + 2mF + Gm^2}{1 + m^2} \) by \( m \), we have

\[
\frac{dr}{dm} = \frac{2 \left( F + (G - E)m - Fm^2 \right)}{(1 + m^2)^2}.
\]

Hence the maximum and minimum values of \( r \) are found by using (10). Now we derive (11). Since we can use (8), we have

\[
\rho^2 - (E + G) \rho + J^2 = \rho^2 - (E + G) \rho + EG - F^2 = (\rho - E)(\rho - G) - F^2.
\]

Now we assume \( m \) satisfies (10), then

\[
\rho - E = \frac{E + 2mF + Gm^2}{1 + m^2} - E = \frac{2mF + (G - E)m^2}{1 + m^2} = mF,
\]

where the last equality was derived from the equation \((G - E)m = F \left( m^2 - 1 \right)\).

Similarly

\[
\rho - G = \frac{E + 2mF + Gm^2}{1 + m^2} - G = \frac{2mF - (G - E)}{1 + m^2} = \frac{F}{m}.
\]

\[
\rho^2 - (E + G) \rho + J^2 = (\rho - E)(\rho - G) - F^2 = 0
\]

the bibliographer: Akio Arimoto