Oscillators with the elliptic function

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1. Statement of the Theorem

The motion of pair of uncoupled oscillators \( \ddot{x}_1 + \omega_1^2 x_1 = 0 \) and \( \ddot{x}_2 + \omega_2^2 x_2 = 0 \) is characterized by the constant total energy \( H = \frac{1}{2} \left[ \dot{x}_1^2 + \dot{x}_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2 \right] \). In fact,
\[
\frac{dH}{dt} = \dot{x}_1 \dot{x}_1 + \dot{x}_2 \dot{x}_2 + \omega_1^2 \dot{x}_1 x_1 + \omega_2^2 \dot{x}_2 x_2 = \dot{x}_1 \left( \dot{x}_1 + \omega_1^2 x_1 \right) + \dot{x}_2 \left( \dot{x}_2 + \omega_2^2 x_2 \right) = 0.
\]
If we consider the constrained motion to move on the circle \( x_1^2 + x_2^2 = 1 \) by imposing an external force \(-f(x)\), that is, \( \ddot{x}_1 + \omega_1^2 x_1 = -f x_1 \) and \( \ddot{x}_2 + \omega_2^2 x_2 = -f x_2 \), \( H \) is still a constant.

\[
\frac{dH}{dt} = \dot{x}_1 \left( \dot{x}_1 + \omega_1^2 x_1 \right) + \dot{x}_2 \left( \dot{x}_2 + \omega_2^2 x_2 \right) = -f \left( \dot{x}_1 x_1 + \dot{x}_2 x_2 \right) = -f \frac{d}{dt} \left( x_1^2 + x_2^2 \right) = 0,
\]
because \( x_1^2 + x_2^2 = 1 \). We now show \( (x_1, x_2) \) will be governed by Weierstrass elliptic function

\[
\wp(z) : \quad x_1 = \sqrt{\frac{b - \omega_1^2}{\omega_2^2 - \omega_1^2}} \quad \text{and} \quad x_2 = \sqrt{\frac{\omega_2^2 - b}{\omega_2^2 - \omega_1^2}} \quad \text{where} \quad 0 < \omega_1^2 < \omega_2^2 \quad \text{and}
\]
\[
b = \wp \left( \sqrt{-1t} + A \right) + B.
\]

2. Proof of the theorem

In this section we prove the following

**Theorem** Constrained system \( \ddot{x}_1 + \omega_1^2 x_1 = -f x_1 \) and \( \ddot{x}_2 + \omega_2^2 x_2 = -f x_2 \), \( 0 < \omega_1^2 < \omega_2^2 \) on the circle \( x_1^2 + x_2^2 = 1 \) has the solution \( x_1 = \sqrt{\frac{b - \omega_1^2}{\omega_2^2 - \omega_1^2}} \) and \( x_2 = \sqrt{\frac{\omega_2^2 - b}{\omega_2^2 - \omega_1^2}} \), where
\[
b = \wp \left( \sqrt{-1t} + A \right) + B.
\]

Let \( b = x_1^2 \omega_2^2 + x_2^2 \omega_1^2 \). Then we easily see \( \omega_1^2 < b < \omega_2^2 \).
In order to prove this Theorem, it is sufficient to show the following lemma, since we might have already known that \( \varphi(z) \) satisfies the differential equation
\[
\varphi^2 = 4(\varphi - \epsilon_1)(\varphi - \epsilon_2)(\varphi - \epsilon_3).
\]

**Lemma** \( \ddot{b}^2 = 4(\omega_2^2 - b)(b - \omega_1^2)[b - (\omega_1^2 + \omega_2^2 - 2H)] \)

**Proof of the lemma**

From equations \( b = x_1^2 \omega_2^2 + x_2^2 \omega_1^2 \) and \( x_1^2 + x_2^2 = 1 \), we have
\[
b - \omega_1^2 = \left( \omega_2^2 - \omega_1^2 \right) x_1^2, \quad \omega_2^2 - b = \left( \omega_2^2 - \omega_1^2 \right) x_2^2.
\]
Since \( 2H = x_1^2 + x_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2 \), we have \( \omega_1^2 + \omega_2^2 - 2H = \omega_1^2 x_2^2 + \omega_2^2 x_1^2 - \left( x_1^2 + x_2^2 \right) \) and \( b - [\omega_1^2 + \omega_2^2 - 2H] = x_1^2 + x_2^2 > 0 \).

Altogether we put them into the multiplication
\[
4(\omega_2^2 - b)(b - \omega_1^2)[b - (\omega_1^2 + \omega_2^2 - 2H)] = 4\left( \omega_2^2 - \omega_1^2 \right)^2 x_1^2 x_2^2 \left( x_1^2 + x_2^2 \right).
\]

On the other hand, differentiating the both sides of \( b - \omega_1^2 = \left( \omega_2^2 - \omega_1^2 \right) x_1^2 \), we get \( \dot{b} = 2\left( \omega_2^2 - \omega_1^2 \right) x_1 \dot{x}_1 \) and
\[
\ddot{b}^2 = 4\left( \omega_2^2 - \omega_1^2 \right)^2 \left( x_1 \dot{x}_1 \right)^2.
\]
To have our result we must only notice that one of the assumptions \( x_1^2 + x_2^2 = 1 \) implies
\[
x_1^2 x_2^2 = (x_1 \dot{x}_1)^2 \text{ and } x_1^2 x_2^2 \left( x_1^2 + x_2^2 \right) = x_1^2 x_2^2 \dot{x}_1^2 + x_1^2 x_2^2 \dot{x}_2^2 = x_1^2 \left( 1 - x_1^2 \right) \dot{x}_2^2 + x_1^2 x_2^2 \dot{x}_1^2 = (x_1 \dot{x}_1)^2.
\]

**Corollary** External force \( f \) for the coupled system \( \ddot{x}_1 + \omega_1^2 x_1 = -f x_1 \), \( \ddot{x}_2 + \omega_2^2 x_2 = -f x_2 \) with \( x_1^2 + x_2^2 = 1 \) is given by \( f = 2H - 2\left( \omega_1^2 x_1^2 + \omega_2^2 x_2^2 \right) \)

Differentiating \( x_1^2 + x_2^2 = 1 \), we have \( x_1 \dot{x}_1 + x_2 \dot{x}_2 = 0 \). Another one more differentiation gives \( \ddot{x}_1^2 + x_1 \ddot{x}_1^2 + \ddot{x}_2^2 + x_2 \ddot{x}_2^2 = 0 \). Substitute \( \ddot{x}_1 = -\omega_1^2 x_1 - f x_1 \) and \( \ddot{x}_2 = -\omega_2^2 x_2 - f x_2 \) into the last formula, we have
\[ x_1^2 + x_1x_2 + x_2^2 + x_2x_3 = x_1^2 + x_2^2 - x_1^2(f + \omega_1^2) - x_2^2(f + \omega_2^2) = 0 \], from which we obtain

\[ f = x_1^2 + x_2^2 - \omega_1^2 x_1^2 - \omega_2^2 x_2^2 = 2H - 2(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) \].

Reference