Power Series Representation of Solution to van der Pol equation

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We are going to discuss the van der Pol differential equation:

\[ \ddot{x} - \varepsilon \left(1 - x^2\right) \dot{x} + x = 0, \]

where \( x \) is a function of \( t \), \( x = x(t) \) and \( \dot{x} \) denotes a differential by \( t \), \( \dot{x} = \frac{dx}{dt} \).

We want to know how solution of (1) is dependent on \( \varepsilon \). Behavior of the solution of (1) can be viewed as a graph in the phase space \((x, y)\), where \( y = \dot{x} \). In anyway (1) is an autonomous system, that is \( t \) does not appear explicitly in (1). So we can represent (1) as the first order differential equation such as

\[ y' - \varepsilon \left(1 - x^2\right)y + x = 0, \]

where \( y \) is a function of \( x \), \( y = y(x) \) and \( y' = \frac{dy}{dx} \).

Now we put \( y = y(x) \) in the power series

\[ y = \sum_{k=0}^{\infty} a_k x^k, \]

then we can write (2) in the following way:

\[ \sum_{k=0}^{\infty} a_k x^k \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j + x = \varepsilon \left(1 - x^2\right) \sum_{k=0}^{\infty} a_k x^k, \]

\[ (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) + x \]

\[ = \varepsilon \left(a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots\right) - \varepsilon \left(a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots\right) \]

We can see from this equation (4), if we take the coefficients of \( x^k \) of the left hand side equal to the ones of the right hand side,

\[ a_0 a_1 = \varepsilon a_0 \]

\[ 2a_0 a_2 + a_1^2 + 1 = \varepsilon a_1 \]

\[ 3a_0 a_3 + 2a_1 a_2 + a_2 a_1 = \varepsilon \left(a_2 - a_0\right) \]
(9) $4a_0a_4 + 3a_1a_3 + 2a_2^2 + a_3a_1 = \varepsilon (a_3 - a_1)$

(10) $5a_0a_5 + 4a_1a_4 + 3a_2a_3 + 2a_3a_2 + a_4a_1 = \varepsilon (a_4 - a_2)$

(11) $6a_0a_6 + 5a_1a_5 + 4a_2a_4 + 3a_3 + 2a_4a_2 + a_5a_1 = \varepsilon (a_5 - a_3)$

In general we would have

(12) $(k + 1)a_0a_{k+1} + ka_1a_k + (k - 1)a_2a_{k-1} + \cdots + 3a_{k-2}a_3 + 2a_{k-1}a_2 + a_ka_1 = \varepsilon (a_k - a_{k-2}),$

for $k = 2, 3, \cdots$. We can determine $a_0, a_1, a_2 \cdots$ in these equations from (6) to (11) successively.

$a_1 = \varepsilon, \ a_2 = -\frac{1}{2a_0}, \ a_3 = \frac{\varepsilon - a_0^2}{3a_0^2}, \ a_4 = -\frac{\varepsilon^2}{4a_0^3} \cdot \frac{1}{8a_0^4}, \ldots$

Hence we have

$$y(x, \varepsilon) = a_0 + \varepsilon x - \frac{x^2}{2a_0} + \varepsilon \frac{1 - a_0^2}{a_0^2} \frac{x^3}{3} - \left( \frac{2\varepsilon^2}{a_0^3} + \frac{1}{a_0^3} \right) \frac{x^4}{8} + \cdots,$$

where $a_0 = y(0, \varepsilon)$.

Can we prove that $y(x)$ is analytic in $x$ and $\varepsilon$ near $x = \varepsilon = 0$?