Solutions to problems in de BRANGES's book

"Hilbert Spaces of Entire Functions"

THEOREM 1 PHRAGMÉN -LINDELÖF Principle.

Assume that $f(z)$ is analytic in the upper half-plane, that $|f(z)|$ has a continuous extension to the closed half-plane, and that

$\liminf_{a\to\infty} a^{-1}\int_0^\pi \log^+\left|f\left(ae^{i\theta}\right)\right|\sin\theta d\theta = 0$. If $|f(z)|$ is bounded by 1 on the real axis, then it is bounded by1 in the upper half-plane.

Corollary Let f is bounded analytic in the upperhalfplane, $|f(z)|\le M$.

If $|f(z)|$ is bounded by 1 on the real axis, then it is bounded by1 in the upper half-plane.

THEOREM2 Let $h(x)$ be a continuous function of real x such that $h(x)\ge 0$ for all real x and

$$\int_{-\infty}^{\infty}\frac{h(t)}{1+t^2}dt < \infty .$$

Then we can define a function $g(z)$ as an integral

$$g(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+z^2}{1+t^2}\frac{h(t)}{t-z}dt+\frac{z}{\pi i}\int_{-\infty}^{\infty}\frac{h(t)}{1+t^2}dt \quad ,$$

which is analytic in the upper half-plane, $\operatorname{Re} g(z)\ge 0$, $|g(z)|$ has a continuous extension to the closed half –plane, and $Re\ g(x)=h(x)$ for all real x .

Proof) Since we easily see that

$$\mathrm{Re}\left\{\frac{1+z^2}{i(t-z)}+\frac{z}{i}\right\}=y\frac{1+t^2}{|t-z|^2}$$, from which we obtain

$$\mathrm{Re}\, g(x+iy)=\frac{y}{\pi}\int\frac{h(t)}{(t-x)^2+y^2}dt=h*P_y(x),$$

where $P_y(x)=\frac{1}{\pi}\frac{y}{x^2+y^2}$ is the Poisson Kernel in the upper half plane..

The conclusions are just derived from the properties of the Poisson kernel.■

Problem 1. Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that $\mathrm{Re}\, f(z)$ has a continuous extension to the closed half-plane and that $h(x)$ is bounded, continuous function of real x such that $0\le h(x)\le \mathrm{Re}\, f(x)$ for all real x . Show that

$$\mathrm{Re}\, f(x+iy)\ge\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h(t)dt}{(t-x)^2+y^2} \quad , \quad \text{for } y>0 \ .$$

Answer) Let $g(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+z^2}{1+t^2}\frac{h(t)}{t-z}dt+\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{h(t)}{1+t^2}dt$ as being defined in Theorem2. Since $\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{dt}{(t-x)^2+y^2}=1$ and $h(x)$ is bounded , say $h(x)\le M$, we have $\mathrm{Re}\, g(x+iy)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h(t)}{(t-x)^2+y^2}dt\le M$. Let's consider

$\Phi(z)=e^{g(z)-f(z)}$. Then we can see $|\Phi(x)|=e^{h(x)-\mathrm{Re}\,f(x)}\le 1$ because $0\le h(x)\le \mathrm{Re}\ f(x)$. Also $|\Phi(z)|=e^{\mathrm{Re}\,g(z)-\mathrm{Re}\,f(z)}\le e^{\mathrm{Re}\,g(z)}\le e^{M}$. By the remark after theorem 1,we have $\left|e^{g(z)-f(z)}\right|=|\Phi(z)|\le 1$ which implies

$\mathrm{Re}\,g(x+iy)\le \mathrm{Re}\,f(x+iy)$, or $\mathrm{Re}\,f(x+iy)\ge \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{h(t)\,dt}{(t-x)^2+y^2}$ for $y>0$, ∎

Problem 2. Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. If $\mathrm{Re}\ f(z)$ has a continuous extension to the closed half-plane,.

Show that there exists a function $g(z)$, which is analytic and has a nonnegative real part in the upper half-plane, such that

$$\mathrm{Re}\,f(x+iy)=\mathrm{Re}\,g(x+iy)+\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,f(t)\,dt}{(t-x)^2+y^2}\text{, for } y>0 .$$

Show that $\mathrm{Re}\,g(z)$ is continuous in the closed half-plane and that $\mathrm{Re}\,g(x)=0$ for any real x.

Answer) Setting $h(t)=\mathrm{Re}\,f(t)$ in Problem 1, we have

$\mathrm{Re}\,f(x+iy)\ge \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Re}\,f(t)\,dt}{(t-x)^2+y^2}$, $y>0$. If we let x=0 and $y=1$, we have $\mathrm{Re}\,f(i)\ge \int_{-\infty}^{\infty}\frac{\mathrm{Re}\,f(t)}{1+t^2}dt$, which is exactly the condition of Theorem 2

$\int_{-\infty}^{\infty} \frac{\mathrm{Re}\, f(t)}{1+t^2} dt < \infty$ for $h(t) = \mathrm{Re}\, f(t)$. Now if we define

$\tilde{f}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1+z^2}{1+t^2} \frac{\mathrm{Re}\, f(t)}{t-z} dt + \frac{z}{\pi i} \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, f(t)}{1+t^2} dt$, we will have

$\mathrm{Re}\, \tilde{f}(x+iy) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, f(t)}{(t-x)^2 + y^2} dt$ and $\lim_{y \to 0} Re\ \tilde{f}(x+iy) = \mathrm{Re}\, f(x)$ for all real x . Hence if we take $g(z) = f(z) - \tilde{f}(z)$, the desired result is obtained. ∎

Problem 3 Let $g(z)$ be a function which is analytic and has nonnegative real part in the upper half plane. Assume that $\mathrm{Re}\, g(z)$ is continuous in the closed half-plane, and that $\mathrm{Re}\, g(x) = 0$ for all real x .

Show that $\mathrm{Re}\, g(x+iy) = py$ where p is a constant.

Answer)