Periodic solutions of Duffing's differential equation with square wave external force

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1. Introduction

The following non-linear differential equation is called Duffing’s differential equation with external force \( F(t) \):

\[
\frac{d^2 x}{dt^2} + px + 2qx^3 = F(t), \quad p, q > 0
\]

under the initial condition: \( x(0) = 0 \), \( \left. \frac{dx}{dt} \right|_{t=0} = a \). When \( F(t) = F(t + \omega) \), we could ask whether (1.1) have a periodic solution \( x = x(t), \ x(t) = x(t + \omega) \). Choy-Tak Taam [1957] proved the existence theorem imposing the assumption \( F(t) = -F(-t) \), \( 0 \leq F(t) \leq M \) (positive constant \( M \)). Taam also discussed the oscillatory properties by checking out the number of zeros of the solution \( x(t) \). In general we cannot describe an explicit solution for (1.1). In this article we impose a further restriction that \( F(t) \) is the square wave function other than being even function, such as

\[
F(t) = \begin{cases} 
-\frac{e}{2}, & t \in \left( k\omega, \frac{k+1}{2}\omega \right) \\
\frac{e}{2}, & t \in \left( \frac{k+1}{2}\omega, (k+1)\omega \right) 
\end{cases}
\]

and will show that there exists a periodic solution of the explicit form. Actually solutions are expressed by the linear fractional transformation of the Jacobi’s elliptic function having parameters \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) which are the four roots of the quartic function \( f(x) = a^2 + cx - px^2 - qx^4 = 0 \). The solutions of this explicit form may enable us to study about oscillatory properties of \( x = x(t) \) in more detail than the Taam’s ones.
Firstly we will give the solution of (1.1) in the interval \( \frac{0}{2} < t \leq \frac{\omega}{2} \). After that we will extend it to the whole time periodically.

2. Preliminaries and some lemmas

Multiply \( \frac{dx}{dt} \) to the both sides of

\[
(2.1) \quad \frac{d^2 x}{dt^2} + px + 2qx^3 = \frac{e}{2}, \quad 0 < t \leq \frac{\omega}{2}
\]

and integrate that, and we have

\[
(2.2) \quad \left( \frac{dx}{dt} \right)^2 = a^2 + cx - px^2 - qx^4, \quad 0 < t \leq \frac{\omega}{2}.
\]

Let \( f(x) \) be the right side of (2.2). Then

\[
(2.3) \quad f(x) = a^2 + cx - px^2 - qx^4 = -q p_1(x) p_2(x),
\]

\[
p_1(x) = (x - \alpha_1)(x - \alpha_2), \quad p_2(x) = (x - \alpha_3)(x - \alpha_4),
\]

where we can set \( \alpha_1 = \bar{\alpha}_2 \) (complex conjugate) and \( \alpha_4 < 0 < \alpha_3 \) because of the sign condition for coefficients of \( f(x) \). The elementary symmetrical polynomials derived from the resolution into factors of \( p_1(x) p_2(x) \) are

\[
s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,
\]

\[
s_2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_1 \alpha_4 + \alpha_2 \alpha_3 + \alpha_2 \alpha_4 + \alpha_3 \alpha_4,
\]

\[
s_3 = \alpha_1 \alpha_2 \alpha_3 + \alpha_1 \alpha_2 \alpha_4 + \alpha_1 \alpha_3 \alpha_4 + \alpha_2 \alpha_3 \alpha_4,
\]

\[
s_4 = \alpha_1 \alpha_2 \alpha_3 \alpha_4,
\]

from which it is seen

\[
(2.4) \quad s_1 = 0, \quad s_2 = \frac{p}{q}, \quad s_3 = \frac{e}{q}, \quad s_4 = -\frac{a^2}{q}.
\]

Lemma 2.1

\[
(2.5) \quad \alpha_3 + \alpha_4 > 0
\]

proof: Since \( s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \) and \( s_1 = 0 \) in (2.4), we have

\[
\alpha_1 + \alpha_2 = -(\alpha_3 + \alpha_4). \quad \text{On the other hand}
\]

\[
s_3 = \alpha_1 \alpha_2 \alpha_3 + \alpha_1 \alpha_2 \alpha_4 + \alpha_1 \alpha_3 \alpha_4 + \alpha_2 \alpha_3 \alpha_4
\]

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\[ a_1a_2 (a_3 + a_4) + (a_1 + a_2)a_3a_4 = (a_1a_2 - a_3a_4)(a_3 + a_4) \]

From \( a_1a_2 = |a_1|^2 > 0 \) and \( a_3 > 0, a_4 < 0 \), we have \( (a_1a_2 - a_3a_4) > 0 \) and \( s_3 = \frac{e}{q} > 0 \).

Hence \( a_3 + a_4 > 0 . \) \[ \]

We rewrite

\[ (2.6) \quad p_1(x) = x^2 + b_1x + c_1, \quad b_1 = -(a_1 + a_2), c_1 = a_1a_2 \]

\[ (2.7) \quad p_2(x) = x^2 + b_2x + c_2, \quad b_2 = -(a_3 + a_4), c_2 = a_3a_4 . \]

Hence from (2.3) we write \( p_1(x)p_2(x) \) in two ways

\[ (2.8) \quad p_1(x)p_2(x) = x^4 + (b_1 + b_2)x^3 + (c_1 + c_2 + b_1b_2)x^2 + (c_1b_2 + c_2b_1)x + c_1c_2 \]

\[ = x^4 + \frac{p}{q}x^2 - \frac{e}{q}x - \frac{a^2}{q} , \]

and comparing coefficients of the formula we have

\[ (2.9) \quad b_1 + b_2 = 0 \]

\[ (2.10) \quad c_1 + c_2 + b_1b_2 = \frac{p}{q} \]

\[ (2.11) \quad c_1b_2 + c_2b_1 = -\frac{e}{q} \]

\[ (2.12) \quad c_1c_2 = -\frac{a^2}{q} . \]

**Lemma 2.2**

\[ (2.13) \quad b_1 > 0, b_2 < 0 \]

\[ (2.14) \quad c_1 + c_2 > 0 \]

\[ (2.15) \quad c_1 - c_2 > 0 \]

**Proof** In (2.7) we can see \( b_2 = -(a_3 + a_4) \) and (2.9) implies \( b_1 = b_2 = a_3 + a_4 \). By Lemma 2.1 we have (2.13). From (2.10) \( c_1 + c_2 = \frac{p}{q} - b_1b_2 = \frac{p}{q} + b_1^2 > 0 \). (2.9) and (2.11) imply \( b_2(c_1 - c_2) = c_1b_2 + c_2b_1 = -\frac{e}{q} < 0 \) and \( b_2 < 0 \) implies \( c_1 - c_2 > 0 . \)

Now we consider the following specified polynomial

\[ (2.16) \quad h(x) = (b_1 - b_2)x^2 + 2(c_1 - c_2)x + b_2c_1 - b_1c_2 . \]
and denote $m, n$ the roots of the quadratic equation $h(x) = 0$ and $D$ be its
discriminant. Then

\[ D = \left( c_1 - c_2 \right)^2 + 2b^2_1 (c_1 + c_2), \]

(2.17) which is found to be positive from Lemma 2.2. By elementary calculus,

\[ m = \frac{-2(c_1 - c_2) + \sqrt{D}}{4b_1} > 0 \]

(2.18) \hspace{1cm} \hspace{1cm} \hspace{1cm}

\[ n = \frac{-2(c_1 - c_2) - \sqrt{D}}{4b_1} < 0. \]

(2.19) \hspace{1cm} \hspace{1cm} \hspace{1cm}

And we have

\[ m + n = \frac{c_1 - c_2}{2} = \frac{\alpha_1 \alpha_2 - \alpha_3 \alpha_4}{\alpha_1 + \alpha_2}, \]

(2.20) \hspace{1cm} \hspace{1cm} \hspace{1cm}

\[ mn = \frac{c_1 + c_2}{2} = -\left( \frac{\alpha_1 \alpha_2 + \alpha_3 \alpha_4}{2} \right) \]

(2.21) \hspace{1cm} \hspace{1cm} \hspace{1cm}

and

\[ m - n = \frac{\sqrt{D}}{2b_1}. \]

(2.22)

**Lemma 2.3**

\[ D = 4\left\{ (c_1 - c_2)^2 + 2b^2_1 (c_1 + c_2) \right\} \]

(2.23)

**Proof** By the direct computation we get from (2.6), (2.7)

\[ p_1(n)p_2(m) - p_1(m)p_2(n) = \frac{D\sqrt{D}}{8b^2_1}, \]

(2.24)

\[ D = 4\left\{ (c_1 - c_2)^2 + 2b^2_1 (c_1 + c_2) \right\} \]

(2.23)

**Proof** By the direct computation we get from (2.6), (2.7)

\[ p_1(n)p_2(m) - p_1(m)p_2(n) \]

\[ = (m - n)\left\{ -2b_1 mn - (c_1 - c_2)(m + n) + b_1 (c_1 + c_2) \right\} \]

\[ = \frac{\sqrt{D}}{2b_1} \left\{ 2b_1 \left( \frac{c_1 + c_2}{2} \right) + (c_1 - c_2) \left( \frac{c_1 - c_2}{b_1} \right) + b_1 (c_1 + c_2) \right\} \]

\[ = \frac{\sqrt{D}}{2b^2_1} \left( 2b^2_1 (c_1 + c_2) + (c_1 - c_2)^2 \right) = \frac{D\sqrt{D}}{8b^2_1}. \]

(2.24)

We easily see that $p_1(n) > 0$ and $p_1(m) > 0$ since $p_1(x) = \left| x - \alpha_1 \right|^2$ for a real
number $x$. We will next prove that $p_2(n) > 0$ but $p_2(m) < 0$. In order to do this, we
must compare the graphs of $h(x)$ and of $p_2(x)$. $h(x)$ is upwards convex parabola which is crossing the $x$-axis at $n < 0$ and $m > 0$. On the other hand $p_2(x)$ is concave up parabola which is crossing the $x$-axis at $\alpha_1 < 0$ and $\alpha_3 > 0$. From this fact we can prove the following lemma.

**Lemma 2.4**

(2.24) $h(\alpha_3) > 0$, $h(\alpha_4) < 0$

(2.25) $p_2(m) < 0$, $p_2(n) > 0$

(2.26) $n < \alpha_4 < 0 < m < \alpha_3$

**Proof**

(2.27) $h(x) = (b_1 - b_2)x^2 + 2(c_1 - c_2)x + b_2c_1 - b_1c_2$

$$= 2(\alpha_3 + \alpha_4)x^2 + 2(\alpha_1\alpha_2 - \alpha_3\alpha_4)x - (\alpha_3 + \alpha_4)(\alpha_1\alpha_2 + \alpha_3\alpha_4)$$

Let $x = \frac{m + ny}{1 + y}$. Then we can have the following lemma.

**Lemma 2.5**

(2.28) $p_1(x) = p_1\left(\frac{m + ny}{1 + y}\right) = \frac{p_1(m) + p_1(n)y^2}{(1 + y)^2}$

(2.29) $p_2(x) = p_2\left(\frac{m + ny}{1 + y}\right) = \frac{p_2(m) + p_2(n)y^2}{(1 + y)^2}$

**Proof**

$$p_1\left(\frac{m + ny}{1 + y}\right) = \frac{(m + ny)^2}{1 + y} + b_1\left(\frac{m + ny}{1 + y}\right) + c_1$$

$$= \frac{(m + ny)^2 + b_1(m + ny)(1 + y) + c_1(1 + y)^2}{(1 + y)^2}$$

$$= \frac{(m^2 + b_1m + c_1)^2 + (2mm + b_1(m + n) + 2c_1)y + (n^2 + b_1n + c_1)^2y^2}{(1 + y)^2}$$

However

(2.20),(2.21) imply $2mm + b_1(m + n) + 2c_1 = 0$. In other words, we have
\[ p_1 \left( \frac{m + ny}{1 + y} \right) = \frac{p_1 \left( m \right) + p_1 \left( n \right)y^2}{\left( 1 + y \right)^2} \]. We can also prove (3.2) in a similar way.

3. Transformation to a differential equation which will give the explicit solution

Let \( y = y(t) \) be the function satisfying \( x(t) = \frac{m + ny(t)}{1 + y(t)} \). Since \( dx = \frac{-m-n}{\left( 1 + y \right)^2} dy \),

\[
\left( \frac{dx}{dt} \right)^2 = -qp_1 \left( x \right) p_2 \left( x \right)
\]

becomes

\[
\left( m - n \right)^2 \left( \frac{dy}{dt} \right)^2 = -q \left\{ p_1 \left( m \right) + p_1 \left( n \right)y^2 \right\} \left\{ p_2 \left( m \right) + p_2 \left( n \right)y^2 \right\}.
\]

We will show \( y(t) \) is a Jacobi's elliptic function. Let \( A^2 = \frac{p_1 \left( m \right)}{p_1 \left( n \right)} \) and \( B^2 = \frac{-p_1 \left( m \right)}{p_1 \left( n \right)} \).

By (2.22) we have

\[
\left( \frac{dy}{dt} \right)^2 = \frac{D}{4b'^2} \left( \frac{dy}{dt} \right)^2 = qp_1 \left( n \right) p_2 \left( n \right) \left\{ A^2 + y^2 \right\} \left\{ B^2 - y^2 \right\}
\]

Here if we transform again \( y = Bz \),

using (2.22) and the relation in Lemma 2.3

\[
\left( A^2 + B^2 \right) p_1 \left( n \right) p_2 \left( n \right) = \frac{D\sqrt{D}}{8b'^2}
\]

we have

\[
\left( \frac{dz}{dt} \right)^2 = \frac{q\sqrt{D}}{2} \left( k'^2 + k^2z^2 \right) \left( 1 - z^2 \right),
\]

where \( k^2 = \frac{A^2}{A^2 + B^2}, \quad k'^2 = 1 - k^2 \).

Thus \( z(t) = cn \left( \sqrt{\frac{q\sqrt{D}}{2}}, t, k \right) \), where \( cn(u, k) \) is the Jacobi's elliptic function and its inverse is defined by \( cn^{-1}(z, k) = \int_0^1 \frac{dz}{\sqrt{\left( 1 - z^2 \right) \left( k'^2 + k^2z^2 \right)}} \). \( cn(t, k) \) has a period \( 4K \),

where \( K = K \left( k \right) \) is the first kind elliptic complete integral

\[
K \left( k \right) = \int_0^\frac{\pi}{2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}.
\]

Hereafter we will try to seek a solution of the form which
depends on the four roots \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( q \).

**Lemma 3.1** \( D = 4p_1(\alpha_3)p_1(\alpha_3) = 4|\alpha_1 - \alpha_3|^2|\alpha_2 - \alpha_4|^2 \)

**Proof**

\[
p_1(\alpha_3)p_1(\alpha_3) = (\alpha_3^2 + b_1\alpha_3 + c_1)(\alpha_4^2 + b_1\alpha_4 + c_1)
\]

\[
= (\alpha_3\alpha_4)^2 + b_1\alpha_3\alpha_4(\alpha_3 + \alpha_4) + c_1\left\{(\alpha_3 + \alpha_4)^2 - 2\alpha_3\alpha_4\right\}
\]

\[
+ b_1^2\alpha_3\alpha_4 + b_1c_1(\alpha_3 + \alpha_4) + c_1^2
\]

\[
= c_2^2 - b_1c_2b_1 + c_1\left(b_2^2 - 2c_2\right) + b_1^2c_2 - b_1c_1b_2 + c_1^2
\]

\[
= (c_1 - c_2)^2 - (b_1 - b_2)(b_2c_1' - b_1c_2'),
\]

which is to be \( \frac{D}{4} \).

We can also write \( B \) in a more simple form.

**Lemma 3.2**

\[
B^2 = \frac{p_1(m)}{p_1(n)} = \frac{(\alpha_3 - m)(\alpha_4 - m)}{(\alpha_3 - n)(\alpha_4 - n)}
\]

can be written by

\[
(3.5) \quad B = \frac{\alpha_3 - m}{\alpha_3 - n} = \frac{m - \alpha_4}{\alpha_4 - n}
\]

**Proof** The second equality in (3.5) comes from the following equality:

\[
(\alpha_3 - m)(\alpha_4 - n) - (m - \alpha_4)(\alpha_3 - n)
\]

\[
= 2mn + 2\alpha_3\alpha_4 - (\alpha_3 + \alpha_4)(m + n)
\]

which is easily found to be zero because of (2.20)(2.21).

**Lemma 3.3**

\[
\alpha_3 = \frac{m - nB}{1 - B}, \quad \alpha_4 = \frac{m + nB}{1 + B}
\]

**Proof** These are direct consequences the following relations derived from (3.5):

\[
\frac{m - nB}{1 - B} = 1 - \frac{\alpha_3 - m}{\alpha_3 - n} = \alpha_3, \quad \frac{m + nB}{1 + B} = 1 + \frac{\alpha_3 - m}{\alpha_3 - n} = \alpha_4
\]
Lemma 3.4.

\[0 < -\frac{m}{n} < B < 1\]

**Proof.** The result is a consequence of the definition of \( B \).

\[1 - B = 1 - \frac{\alpha_3 - m}{\alpha_3 - n} = \frac{m - n}{\alpha_3 - n} > 0, \quad B + \frac{m}{n} = \frac{m - \alpha_4}{\alpha_4 - n} + \frac{m}{n} = \frac{\alpha_4 (m - n)}{n(\alpha_4 - n)} > 0\]

We can now give the explicit expression to the differential equation

\[
\frac{d^2x}{dt^2} + px + 2qx^3 = \frac{e}{2}, \quad 0 < t \leq \frac{\omega}{2}, \quad p, q, e > 0.
\]

with the initial condition: \( x(0) = 0 \) and \( \frac{dx}{dt} \bigg|_{t=0} = a \).

We saw \( z(t) \) is the Jacobi’s cn function above, where \( x = \frac{m + ny}{1 + y} \) and \( y = Bz \). But we must notice that the time \( t \) in \( z(t) \) is not the same as the time \( t \) in \( x(t) \). The difference \( t_0 \) of these is determined by the following lemma.

**Lemma 3.5** Let \( K = K(k) \) is the first kind elliptic complete integral.

\[y(t) = \text{Bcn}(\Omega(t + t_0), k), \quad \Omega = \sqrt{\frac{1}{2}} D^3,\]

where

\[0 < t_0 < \frac{K}{\Omega} \text{ if } a > 0, \quad -\frac{K}{\Omega} < t_0 < 0 \text{ if } a < 0\]

and \( t_0 \) can be found uniquely in each intervals.

**Proof** Initial condition \( x(0) = 0 \) yields \( y(0) = -\frac{m}{n} \) since \( x(t) = \frac{m + ny(t)}{1 + y(t)} \). From Lemma 3.4 \( Bz(\Omega t_0, k) = y(0) = -\frac{m}{n} < B \), and this relation means we can find two possible \( t_0 \) in the interval \( \left[ -\frac{K}{\Omega}, \frac{K}{\Omega} \right] \). However we see \( \dot{x}(0) = -\frac{m - n}{(1 + y^2(0))} \dot{y}(0) \) and \( \dot{y}(0) = -B\Omega \text{sn}(\Omega t_0, k) \text{dn}(\Omega t_0, k) \). Combining these fact and another initial condition \( \frac{dx}{dt} \bigg|_{t=0} = a \), we have the desired result.
We can now give the explicit expression to the differential equation

Theorem 3.6. Non-linear differential equation

\[
\frac{d^2 x}{dt^2} + px + 2qx^3 = \frac{e}{2}, \quad 0 < t \leq \frac{\omega}{2}, \quad p,q,e > 0,
\]

with the initial condition: \( x(0) = 0 \) and \( \frac{dx}{dt} \bigg|_{t=0} = a \)

has the solution

\[
x(t) = \frac{m + ny(t)}{1 + y(t)}, \quad 0 < t \leq \frac{\omega}{2}
\]

where

\[
y(t) = Bcn(\Omega(t + t_0)k), \quad \Omega = \sqrt{\frac{2}{\omega}} D^2,
\]

\[
B = \frac{\alpha_3 - m}{\alpha_3 - n} = m - \frac{\alpha_4}{\alpha_4 - n} > 0, \quad m = \frac{-2(c_1 - c_2) + \sqrt{D}}{4b_1} > 0
\]

\[
n = \frac{-2(c_1 - c_2) - \sqrt{D}}{4b_1} < 0
\]

\[
D = 4p_1(\alpha_3)p_1(\alpha_3) = 4|\alpha_1 - \alpha_3|^2|\alpha_2 - \alpha_3|^2,
\]

\( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) are roots of the quadratic equation

\[
a + ex - px^2 - qx^4 = 0
\]

4. Construction of periodic solution \( x = x(t) \) over \(-\infty < t < \infty\)

In this section we would make an extension of the solution \( x = x(t) \) defined on

\( 0 \leq t < \frac{\omega}{2} \) to the one defined over the entire time domain \(-\infty < t < \infty\). Although solutions constructed are surely periodic \( x(t) = x(t + \omega) \), their oscillation patterns can be changed in many ways. First, we will give the existence theorem of the extended solutions.

Theorem 4.1. Let \( 0 < \omega \leq \frac{8K}{\Omega} \). Then

\[
(4.1) \quad \frac{d^2 x}{dt^2} + px + 2qx^3 = F(t), \quad -\infty < t < \infty
\]
\[ F(t) = \begin{cases} 
\frac{e}{2}, & t \in \left[ \frac{k\omega}{2}, \left( k + \frac{1}{2} \right) \omega \right) \\
-\frac{e}{2}, & t \in \left( \left( k + \frac{1}{2} \right) \omega, (k + 1)\omega \right] 
\end{cases} \]

\( k = 0, \pm 1, \pm 2, \ldots \)

has a periodic solution \( x(t) = x(t + \omega) \) satisfying \( x(0) = 0 \), \( \frac{dx}{dt} \bigg|_{t=0} = a \).

**Proof**  

The function \( F(t) \) is odd. Hence we consider the next differential equation

\[ \frac{d^2x}{dt^2} + px + 2qx^3 = -\frac{e}{2} \frac{\omega}{2} \leq t < \omega. \]

Here we firstly show that if we assume \( x\left(\frac{\omega}{2}\right) = 0 \) and \( \dot{x}\left(\frac{\omega}{2}\right) = -a \), we can extend the solution in Theorem 3.6 to \( \frac{\omega}{2} \leq t < \omega \). And then we will show that \( x\left(\frac{\omega}{2}\right) = 0 \) and \( \dot{x}\left(\frac{\omega}{2}\right) = -a \). Now suppose we have \( x(t) \) satisfying (3.6). Let \( w(t) = -x\left(t + \frac{\omega}{2}\right), 0 \leq t < \frac{\omega}{2} \). Then \( w(t) \) satisfies (3.7) including the initial condition.

In other words \( x(t) = w(t) = -x\left(t + \frac{\omega}{2}\right), 0 \leq t < \frac{\omega}{2} \). In this way we have the extended solution \( x = w(t), 0 \leq t < \omega \). Furthermore we extends this \( x = w(t), 0 \leq t < \omega \) to \( x = x(t), -\infty < t < \infty \), so as to satisfy \( x(-t) = -x(t) \). It is remaining to see that

\( x\left(\frac{\omega}{2}\right) = 0 \) and \( \dot{x}\left(\frac{\omega}{2}\right) = -a \). From the relation, \( x(t) = \frac{m + ny(t)}{1 + y(t)} \), these equalities equivalent only to \( y\left(\frac{\omega}{2}\right) = y(0) \). Because \( y\left(\frac{\omega}{2}\right) = y(0) \) implies \( x\left(\frac{\omega}{2}\right) = 0 \) and \( \dot{x}\left(\frac{\omega}{2}\right) = -a \) considering the relation of \( \dot{x}(t) = \frac{(n - m)\dot{y}(t)}{1 + y^2(t)} \), \( n - m < 0 \). But if we take

\[ \frac{\omega}{2} = \frac{4K - 2L_w}{\Omega} \text{ in } y(t) = Bcn(\Omega(t + t_w), k), \text{ we get } y(0) = Bcn(\Omega t_w, k) = -\frac{m}{n} \text{ and } \]

\[ y\left(\frac{\omega}{2}\right) = Bcn\left(\Omega\left[\frac{4K - 2L_w}{\Omega} + t_0\right], k\right) = Bcn(\Omega t_w, k) = -\frac{m}{n}. \]

Remark: We should notice that in constructing \( x(t) \), we don’t use the whole of
\( y(t) = Bcn(\Omega(t + t_0), k), \ 0 \leq t < \frac{4K}{\Omega}, \) but only the part of \( y(t) \), in \( 0 \leq t < \frac{\omega}{2} \). So we must have imposed the condition \( \omega \leq \frac{8K}{\Omega} \).

Lastly, we are going to depict the result of the Theorem in the following illustrations.
The diagrams show the behavior of functions $y(t)$ and $x(t)$ over time $t$. The diagrams are labeled with various constants and time intervals.

- The upper diagram has a note $\text{图 } 2 (a>0)$ indicating a specific condition or context for these functions.
- The lower diagram is labeled $\text{图 } 2 (a>0)$, suggesting it is a continuation or a different set of functions under the same condition.

The diagrams appear to illustrate the phase and amplitude changes of the functions over time, with specific points of interest marked, such as $2t_0$. The notation and labels suggest a study or analysis of these functions under certain conditions.
In a similar but a bit different way, we obtain for $a < 0$, 