Def: $S, \bar{S}$ are isometric if there is a diffeomorphism $f$ which takes each curve on $S$ to a curve on $\bar{S}$ are of the same length. Such an $f$ is called an isometry.

Patch of a surface is the image of $X: U \to S$ for a local chart $X$.

Theorem: Two patches of surfaces $S$ and $\bar{S}$ are isometric if they can be parametrized by $X: U \to S$ and $\bar{X}: \bar{U} \to \bar{S}$ in a way that the first fundamental forms coincide.
Pf) \(\leftarrow\) \(\Rightarrow\) \(\sim\)

\[ f = X^{-1} \circ \tilde{X} \quad \text{を考えよ。} \quad f \text{はisometry} \]

\((=)\) isometry

\[ f : \tilde{S} \rightarrow \tilde{S} \]

Start with any local chart

\( \tilde{X} : \tilde{U} \rightarrow \tilde{S} \) covering the patch on \( \tilde{S} \)

Define \( \tilde{\tilde{X}} : \tilde{U} \rightarrow \tilde{S} \) such that

\[ \tilde{\tilde{X}} = f \circ \tilde{X} . \]

Now \( E, F, G \) define by \( \tilde{X} \).
Since this is isometry,

\[ A(\mathcal{U}(\mathfrak{H}), \mathcal{V}(\mathfrak{H})) \text{ is an isometry} \]

\[ \int_0^1 (E u'^2 + 2F u' v' + G v'^2)^{1/2} \, dt \]

\[ = \int_0^1 (E u'^2 + 2F u' v' + G v'^2)^{1/2} \, dt \]

Fix \((u_0, v_0) \in \mathcal{U}, \text{ look at} \)

\((u_0 + t, v_0), \text{ then we have} \)

\[ \int_0^1 E^{1/2} \, dt = \int_0^1 \frac{E^{1/2}}{E} \, dt \]

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \mathfrak{h}(t) \, dt = \mathfrak{h}(0) \]

\[ E_{\chi} = \mathcal{E} \chi \text{ on } E = \mathcal{E} \]

\[ \mathbb{E} \mathbb{G} = \mathcal{G} \]

\[ \int_0^1 (E u'^2 + 2F u' v' + G v'^2)^{1/2} \, dt \]

\[ = \int_0^1 (E u'^2 + 2F u' v' + G v'^2)^{1/2} \, dt \]

\[ \mathcal{F} = \mathcal{F} \]
2) \( C = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 2^2 \} \)

In the plane 

\( (u, v) = (r, \theta) \) 

\( x(u, v) = \left( \frac{ru}{\sqrt{2}} \cos \left( v \sqrt{2} \right), \frac{ru}{\sqrt{2}} \sin \left( v \sqrt{2} \right), \frac{u}{\sqrt{2}} \right) \) 

\( \tilde{x}(u, v) = (u \cos v, u \sin v, 0) \)
Def. $X : U \to S$ is called
1) conformal if it preserves angles, that is
whenever $t \to (u(t), v(t))$, $t \to (\tilde{u}(t), \tilde{v}(t))$ that intersect at $t = 0$ and on angle $\theta$, their
images $X(u(t), v(t))$,
$X(\tilde{u}(t), \tilde{v}(t))$ form the same angle.
2) area preserving if $\text{Area}(X(U)) = \text{Area}(V)$, $\forall V \subset U$.

Proposition: $X$ is conformal iff
$E = G$ and $F = 0$
2) $X$ is area preserving iff $E G - F^2 = 1$

1) and 2) $X$ isometric if $\delta = 0$. 

\[ \cos \theta = \frac{(u'v' + v'u')}{(u'^2 + v'^2)^{1/2}(u^2 + v^2)^{1/2}} \]

\[ \ddot{x} = u'x_u + v'x_v \]
\[ \dddot{x} = u''x_u + v''x_v \]

\[ \cos \varphi = \frac{\ddot{x}}{\dddot{x}} = u''u' \cdot E \cdot \frac{(u'v' + v'u')}{(u'^2 + v'^2)^{1/2}(u^2 + v^2)^{1/2}} \]

Let \( u(t) = t, \) \( v(t) = 0, \) \( \dot{u}(t) = 0, \) \( \dot{v}(t) = 0, \)

\[ \theta = \frac{\pi}{2} \]
\[ \cos \theta = 0, \quad \cos \varphi = \frac{F}{\sqrt{EG}} \Rightarrow F = 0 \]

\( u(t) = v(t) = \dot{u}(t) = x, \quad \ddot{u}(t) = 0 \) and \( v(t) = 0. \)

\[ 0 = EG \]
Stereographic projection \[ \pi : \mathbb{R}^2 \to S^2 \]

Bézout's theorem

\[ (x, y) \mapsto \left( \frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \]

\[ N = (0, 0, 1), \]
\[ N + t(p - N) = \left( \frac{t x}{t^2 + x^2 + y^2}, \frac{t y}{t^2 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{t^2 + x^2 + y^2} \right) \]
\[ (tx)^2 + (ty)^2 + (1 - t)^2 = 1 \]
\[ t^2 (x^2 + y^2) + (-2t + x^2) = 0 \]
\[ t^2 (x^2 + y^2 + 1) = 2t \]

\[ t = \frac{2}{x^2 + y^2 + 1} \]

\[ N + t(N - N) = \left( \frac{zx}{1 + x^2 + y^2}, \frac{zy}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right) \]
Theorem: The Gaussian curvature is invariant under isometries. That is, if $f: S \to S'$ is an isometry with Gaussian curvatures $K, K'$, then $K(p) = K'(f(p))$ for all $p \in S$.

Proof:

Take $X: U \to S'$ chart. Isometries are isometric with respect to the Gram-Schmidt orthogonalization. Hence, $X^* \in \mathbb{R}^2$ and $X^* \cdot X^* = 1$. Thus, the Gaussian curvatures are invariant.
\[
\frac{\partial e_i}{\partial u} = e_i, \quad \frac{\partial e_i}{\partial \nu} = e_i, \quad e_i \cdot e_i = 1
\]

\[
e_i, u = x_1, e_2 + \lambda_1 N
\]

\[
e_i, \nu = \lambda_2 e_2 + \lambda_2 N
\]

\[
e_2, u = [-\lambda_1] e_1 + \mu_1 N
\]

\[
e_2, \nu = -\lambda_2 e_1 + \mu_2 N
\]

\[
\langle e_i, e_2 \rangle = 0
\]

\[
\langle e_1, u, e_2 \rangle + \langle e_1, e_2, u \rangle = 0
\]

\[
\langle e_1, u, e_2, \nu \rangle = \lambda_1 \mu_2 - \lambda_2 \mu_1
\]

\[
= \frac{\partial \lambda_1}{\partial u} - \frac{\partial \lambda_2}{\partial u} = \frac{e_2 - f^2}{E_G - F^2}
\]

\[
\langle e_i, u, e_2, \nu \rangle = \langle d_1 e_2 + \lambda_1 N, -d_2 e_1 + \mu_2 N \rangle = \lambda_1 \mu_2
\]

\[
\langle e_1, u, e_2, u \rangle = \langle d_2 e_2 + \mu_2 N, -d_1 e_1 + \mu_1 N \rangle = \lambda_2 \mu_1
\]
\[
\langle e_1, u, e_2 \rangle = \lambda_1
\]

\[
\frac{\partial \angle_1}{\partial u} = \frac{\partial}{\partial v} \langle e_1, u, e_2 \rangle = \langle e_1, u, e_2 \rangle + \langle e_1, e_2, u \rangle
\]

\[
\frac{\partial \angle_2}{\partial u} = \frac{\partial}{\partial u} \langle e_1, u, e_2 \rangle = \langle e_1, u, e_2 \rangle + \langle e_1, e_2, u \rangle
\]

\[
\frac{\partial \angle_1}{\partial v} - \frac{\partial \angle_2}{\partial u} = \langle e_1, u, e_2, v \rangle - \langle e_1, u, e_2, u \rangle
\]

Observations:

\[
\frac{EG - F^2}{EG - F^2} = \langle Nu \times Nu, N \rangle
\]

\[
\langle Nu \times Nu, N \rangle = \langle Nu \times Nu, e_1 \times e_2 \rangle
\]

\[
= \langle Nu, e_1 \rangle \langle Nu, e_2 \rangle - \langle Nu, e_1 \rangle \langle Nu, e_2 \rangle
\]

\[
= \langle Nu, e_1 \rangle \langle Nu, e_2 \rangle - \langle Nu, e_2 \rangle \langle Nu, e_1 \rangle
\]

\[
= \lambda_1 \mu_2 - \mu_1 \lambda_2
\]