Asymptotic periodic solution on the stable and the unstable manifolds for non-homogeneous differential systems

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1. Introduction
Let us consider the linear differential equation with constant coefficients:
(1.1)  \( \ddot{x} + a\dot{x} + bx = f, \ t \geq 0 \)
where \( x = x(t), \ \dot{x}(t) = \frac{dx(t)}{dt}, \ \ddot{x}(t) = \frac{d^2x(t)}{dt^2}, \ a, b \in \mathbb{R} \) and \( f = f(t) \) is an integrable function. Introduce new variables \( u_1(t) = x(t), \ u_2(t) = \dot{x}(t), \) then we can transform (1.1) into the matrix form:
(1.2)  \( \begin{bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} u(t) + \begin{bmatrix} g(t) \end{bmatrix}, \) with \( u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \ A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}, \)
\( g(t) = \begin{bmatrix} \int_0^t f(t') \, dt' \end{bmatrix}. \) We take henceforce the norm in \( \mathbb{R}^2 \)
\( \left\| \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \right\| = \max \{|\xi_1|, |\xi_2|\}. \) Let \( \lambda_1, \lambda_2 \) be denote the roots of \( \lambda^2 + a\lambda + b = 0 \) and suppose \( \lambda_1 > 0, \lambda_2 < 0 \) or \( \lambda_1 < 0, \lambda_2 > 0 \). Due to the Jordan decomposition theorem, we can write \( A = VJV^{-1} \) for Jordan matrix \( J \) and \( e^{tA} = Ve^{tJ}V^{-1}. \) For \( A \), the Jordan is \( J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \)

(1.3)  \( V = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}, V^{-1} = \begin{bmatrix} 1 & 1 \\ \lambda_1 - \lambda_2 & \lambda_1 \end{bmatrix}, e^{tJ} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}, \)

According to (1.2),

(1.4)  \( e^{tA} \frac{d}{dt} (e^{-tA}u(t)) = \dot{u}(t) - Au(t) = g(t) \)

Hence we can write the solution of (1.2) in the following form:

(1.5)  \( u(t) = e^{tA}u(0) + e^{tA} \int_0^t e^{-sA}g(s) \, ds. \)

If \( u(t) \) is periodic with the period \( \omega, \) it must be the unique one.
(1.6) \( u^* (t) = e^{\omega t} \left( 1 - e^{\omega t} \right)^{-1} \int_{t}^{t+\omega} e^{-\omega s} g(s) ds \).

Actually, from (1.4) we can write the relation \( u(t) = u(t + \omega) \),

\[
e^{\omega t} \left[ u(0) + \int_{0}^{t} e^{-\omega s} g(s) ds \right] = e^{(t+\omega)\omega} \left[ u(0) + \int_{0}^{t} e^{-\omega s} g(s) ds + \int_{t}^{t+\omega} e^{-\omega s} g(s) ds \right].
\]

We will denote \( u = u^* \) if it satisfies \( u(t) = u(t + \omega) \). Then from (1.5)

\[
u^* (t) = e^{\omega t} u^* (t) + e^{(t+\omega)\omega} \int_{t}^{t+\omega} e^{-\omega s} g(s) ds,
\]

from which we have (1.5). From (1.7) \( u = u^* \) has another different expression from (1.6)

\[
(1.8) \quad u^* (t) = e^{\omega t} \left[ u^* (0) + \int_{0}^{t} e^{-\omega s} g(s) ds \right]
\]

2. Asymptotic-periodic condition

We will proceed to get the asymptotically periodic solution \( u = u^a \) in the sense

\[
(2.1) \quad \lim_{t \to \infty} |u^a (t) - u^* (t)| = 0,
\]

where \( u^* (t) = e^{\omega t} \left( 1 - e^{\omega t} \right)^{-1} \int_{t}^{t+\omega} e^{-\omega s} g(s) ds \).

Considering the difference of (1.5) and (1.8), we have

\[
(2.2) \quad u(t) - u^* (t) = e^{\omega t} \left( u(0) - u^* (0) \right).
\]

Since we have assumed \( \lambda > 0 \), \( u(t) \) might tend to infinity as \( t \) goes to infinity.

On the other hand \( u^* (t) \) is periodic and continuous function of \( t \) as we can see from (1.6). Indeed we have assumed \( f(t) \) is integrable to assure this. The general solution must be restricted to satisfy (2.1). We will show this can be if we take the initial condition from the real line (the stable manifold).
Theorem 2.1 Suppose $\lambda_1 > 0, \lambda_2 < 0$. If the initial condition

$u_1^*(0) = x(0), u_2^*(0) = \dot{x}(0)$ satisfies $-\lambda_2 u_1(0) + u_2(0) = -\lambda_2 u_1^*(0) + u_2^*(0),$

then

We have $\lim_{t \to \infty} \|u^*(t) - u^*(t)\| = 0$. Here $u_1^*(0), u_2^*(0)$ are components of

\[ u^*(0) = e^{\omega A} \left(1 - e^{\omega A}\right)^{-1} \int_0^\infty e^{-\omega s} g(s) ds. \]

Proof) From (2.2)

\[ (2.3) V^{-1}(u(t) - u^*(t)) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} V^{-1}(u(0) - u^*(0)). \]

And from (1.3) we have

\[ (2.4) \begin{vmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{vmatrix} (u(t) - u^*(t)) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{vmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{vmatrix} (u(0) - u^*(0)), \]

\[ (2.5) \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} (-\lambda_2 \xi + 1) = \begin{bmatrix} e^{\lambda t} (-\lambda_2 + \eta) \\ e^{\lambda t} (\lambda_1 - \eta) \end{bmatrix}. \]

Theorem 2.2 Suppose $\lambda_2 > 0, \lambda_1 < 0$. If the initial condition $u_1^*(0) = x(0), u_2^*(0) = \dot{x}(0)$ satisfies $\lambda_1 u_1(0) - u_2(0) = \lambda_1 u_1^*(0) - u_2^*(0).$ Here $u_1^*(0), u_2^*(0)$ are components of

\[ u^*(0) = e^{\omega A} \left(1 - e^{\omega A}\right)^{-1} \int_0^\infty e^{-\omega s} g(s) ds. \]

Proof) Almost the same as Theorem 2.1. Also refer (2.4)
We call the stable manifold \( W_i = \left\{ (\xi, \eta) \in \mathbb{R}^2 : \lambda_i \xi - \eta = \lambda_i u_i^*(0) - u_i^*(0), \lambda_i < 0 \right\} \).

In theorem 2.1 if we take the initial condition on this stable manifold \( W_i \), we can obtain an asymptotic periodic solution \( u = u^a \).