Let $C$ be the unit circle. Asymptotic expansions for Toeplitz determinants are given corresponding to a family of symbols (1)

$$f(z) = (z - e^{\alpha + \beta} (z - e^{-t})^{\alpha - \beta} z^{-\alpha + \beta} e^{-i\pi(\alpha + \beta)} e^V(z),$$

where $\alpha \pm \beta \neq -1, -2, \cdots, t > 0$ and $V(z) = \sum_{k=-\infty}^{\infty} V_k z^k$ is an analytic function of $z \in C$. For $t > 0$, these symbols are regular so that the determinants obey the usual Szegos strong limit theorem. However, if $t = 0$, the symbol is of the form (2)

$$f(z) = |z - 1|^{2\alpha} z^{\beta} e^{-i\pi \beta} e^V(z)$$

which possesses a Fisher-Hartwig singularity. Letting $t \to 0$ in (1), we will have a transition between two different types of asymptotic behavior for Toeplitz determinants. This transition is shown to be described by the Painlevé differential equation of type V.

Let $f_j = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} d\theta$. If we assume $\ln f \in L^1(C)$, $\sum_{k=-\infty}^{\infty} |k| |(\ln f)_k| < +\infty$ and $(\ln f)_k = \frac{1}{2\pi} 2\pi \int_0^{2\pi} \ln f(e^{i\theta}) e^{-ik\theta} d\theta$, then we can prove the usual strong Szegos limit theorem $\ln D_n = \frac{1}{2\pi} \int_0^{2\pi} \ln f(e^{i\theta}) d\theta + \sum_{k=1}^{\infty} k(\ln f)_k (\ln f)_{-k} + o(1)$ as $n \to \infty, t > 0$. If $f(z)$ in (1), then this Szegos limit theorem becomes

$$\ln D_n(t) = nt (\alpha + \beta) + nV_0 + \sum_{k=1}^{\infty} k \left[V_k - (\alpha + \beta) e^{-tk} \right] V_{-k} - (\alpha - \beta) \frac{e^{-tk}}{k}$$

$$+ \ln \frac{G(\alpha + \beta)G(\alpha - \beta)}{G(1 + 2\alpha)} + \Omega(2nt) + o(1),$$

as $n \to \infty, t > 0$. Here $G(z)$ is called the Barnes G-function and we have an expression $\Omega(2nt) = \int_0^{2nt} \frac{\sigma(x) - x^2 + \beta^2}{x} dx + (\alpha^2 - \beta^2) \ln 2nt$. We find that $\sigma(x)$ in this integral is a particular solution to the Painlevé differential equation of type V:

$$\left(x \frac{d^2\sigma}{dx^2}\right)^2 = \left(\sigma - x \frac{d\sigma}{dx} + 2 \left(\frac{d\sigma}{dx}\right)^2 + 2\alpha \frac{d\sigma}{dx}\right)^2 -$$
\[4 \left( \frac{d\sigma}{dx} \right)^2 \left( \frac{d\sigma}{dx} + \alpha + \beta \right) \left( \frac{d\sigma}{dx} + \alpha - \beta \right).\]

Using the asymptotic properties of the solution \( \sigma(x) \), we can calculate a transition when \( t \to 0 \). It is also explained how this transition between Szego weight and Fisher-Hartwig weight arises in view of two-dimensional Ising model and one-dimensional Heisenberg spin chains.