An integral equation satisfied by the solutions of the van der Pol equation
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July 18, 2007

Our van der Pol equation is

\( \ddot{y} - \varepsilon \left(1 - y^2\right) \dot{y} + y = 0 \)

where \( \dot{y} = \frac{dy}{dt} \) and \( \ddot{y} = \frac{d^2y}{dt^2} \).

We can write this in a matrix form as

\[ \dot{x} = Ax - \varepsilon \xi, \]

where

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}, \quad x = \begin{bmatrix} y \\ \eta \end{bmatrix}, \quad \xi = \begin{bmatrix} 0 \\ y^2 \eta \end{bmatrix}. \]

Here \( A \) has two eigen values \( r_+, r_- \),

where \( r_+ = \frac{\varepsilon + \sqrt{4 - \varepsilon^2} i}{2} \) and \( r_- = \frac{\varepsilon - \sqrt{4 - \varepsilon^2} i}{2} \).

\( A \) has the spectral representation

\[ A = r_+ P_1 + r_- P_2, \]

\[ I = P_1 + P_2, \quad P_1 P_2 = P_2 P_1 = 0. \]

Indeed

\[ P_1 = \frac{1}{r_+ - r_-} \begin{bmatrix} -r_- & 1 \\ 1 & \varepsilon - r_+ \end{bmatrix}, \quad P_2 = \frac{1}{r_+ - r_-} \begin{bmatrix} r_+ & -1 \\ 1 & r_+ - \varepsilon \end{bmatrix} = \frac{1}{r_+ - r_-} (r_+ - A), \]

which enable us to write the exponential function of \( A \) as

\[ e^{At} = e^{r_+ t} P_1 + e^{r_- t} P_2 = \frac{1}{r_+ - r_-} \left\{ \left( e^{r_+ t} - e^{r_- t} \right) A + \left( r_+ e^{r_+ t} - r_- e^{r_- t} \right) \right\}. \]

Using this formula we can make product:

\[ e^{At} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{e^{r_+ t} - e^{r_- t}}{r_+ - r_-} \begin{bmatrix} x_2 \\ -x_1 + \varepsilon x_2 \end{bmatrix} + \frac{r_+ e^{r_+ t} - r_- e^{r_- t}}{r_+ - r_-} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \]

Then we can now write the integral equation of the solution for the van der Pol equation (1).
Generally, we can write

\[ x(t) = e^{At}x(0) - \mathcal{E}\int_0^t e^{A(t-s)}s(s)\,ds , \]

from which we can easily see that

\[ y(t) = \frac{r_+ e^{r_+ t} - r_- e^{r_- t}}{r_+ - r_-} y(0) + \frac{e^{r_+ t} - e^{r_- t}}{r_+ - r_-} \dot{y}(0) - \mathcal{E}\int_0^t \frac{e^{r_+(t-s)} - e^{r_-(t-s)}}{r_+ - r_-} y^2(s) \dot{y}(s)\,ds \]

and

\[ \dot{y}(t) = -\frac{e^{r_+ t} - e^{r_- t}}{r_+ - r_-} y(0) + \frac{(r_+ + \mathcal{E}) e^{r_+ t} - (r_- + \mathcal{E}) e^{r_- t}}{r_+ - r_-} \dot{y}(0) - \mathcal{E}\int_0^t \frac{(r_+ + \mathcal{E}) e^{r_+(t-s)} - (r_- + \mathcal{E}) e^{r_-(t-s)}}{r_+ - r_-} y^2(s) \dot{y}(s)\,ds \]