m-canonical measures for moment sequences

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Let

\[ \mathcal{P} = \{ (x - i)^n | n \geq 0 \} \]

and

\[ \mathcal{M}_0 \equiv \overline{\mathcal{P}}^{L^2(\mu)} \]

Theorem 1

\[ (x - i)^{-1} \in \mathcal{M}_0 = \overline{\mathcal{P}}^{L^2(\mu)} \]

\[ \iff \]

\[ \mathcal{M}_0 = L^2(\mu) \]

Proof

\((\Leftarrow)\) It is obvious.

\((\Rightarrow)\) In the first, we must prove that \((x - i)^{-2} \in \mathcal{M}_0\). We can choose \(p(x) \in \mathcal{P}\) such that

\[ \int \left| \frac{1}{(x - i)^2} - \frac{p(x)}{x - i} \right|^2 d\mu \leq \int \left| \frac{1}{x - i} - p(x) \right|^2 d\mu < \varepsilon \]

Now if we write

\[ p(x) = (x - i)r(x) + c, \]

then

\[ \frac{p(x)}{x - i} = r(x) + \frac{c}{x - i}, \quad r(x) \in \mathcal{P}. \]

Hence by virtue of our assumption we can choose \(q(x) \in \mathcal{P}\) such that

\[ \int \left| \frac{p(x)}{x - i} - q(x) \right|^2 d\mu < \varepsilon. \]

By using the polynomial \(q(x)\), we can approximate \((x - i)^{-2}\):

\[ \left( \int \left| \frac{1}{(x - i)^2} - q(x) \right|^2 d\mu \right)^{\frac{1}{2}} = \left( \int \left| \frac{1}{(x - i)^2} - \frac{p(x)}{x - i} + \frac{p(x)}{x - i} - q(x) \right|^2 d\mu \right)^{\frac{1}{2}} \]
\[
\frac{1}{(x-i)^2} \in \mathcal{M}_0.
\]
Similarly, we can see for \( n = 3, 4, \ldots \) that
\[
\frac{1}{(x-i)^n} \in \mathcal{M}_0.
\]

Next we shall show that
\[
\left\{ \frac{1}{(x-i)^n} \right\}_{n=0, \pm 1, \pm 2, \ldots}^{L^2(\mu)} = L^2(\mu)
\]
However we only have to show that
\[
\left\{ \frac{1}{(x-i)^n} \right\}_{n=0, 1, 2, \ldots}^{L^2(\mu)} = L^2(\mu)
\]

Now if we assume that for \( f \in L^2(\mu) \),
\[
\int \frac{f(x)}{(x-i)^{n+1}} d\mu = 0, \quad n = 0, 1, 2, \ldots,
\]
we must prove \( f(x) = 0 \). Indeed we expand an analytic function as Taylor series:
\[
\int \frac{f(x)}{x-z} d\mu(x) = \sum_{n=0}^{\infty} \int \frac{f(x)}{(x-i)^{n+1}} d\mu(x)(z-i)^n = 0
\]
in the neighbourhood of \( z = i \), from which we have
\[
\int \frac{f(x)}{x-z} d\mu(x) = 0, \quad z \in \mathbb{R}^+,
\]
and so we have \( f(x) = 0 \), a.e. \( \mu \). In a similar way,

**Theorem 2**

\[
(x-i)^{-m-1} \in \left\{ (x-i)^n | n \geq -m \right\}^{L^2(\mu)}
\]

\[\iff\]

\[
(x-i)^n | n \geq -m \right\}^{L^2(\mu)} = L^2(\mu)
\]

**Theorem 3** \( \mu \) is \( m \)-canonical

\[\iff\]

\[
(x-i)^{-m-1} \in \left\{ (x-i)^n | n \geq -m \right\}^{L^2(\mu)}
\]

and

\[
(x-i)^{-m-2} \notin \left\{ (x-i)^n | n \geq -m-1 \right\}^{L^2(\mu)}
\]