A function whose Maclaurin series converges everywhere but represents the function at only one point.

(Counter examples in analysis, Gelbaum,B.R.;Olmsted,J.M.H. ,Dover, p.68 )

The function

(1) \[ f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \]

is infinitely differentiable, all of its derivatives at \( x = 0 \) being equal to 0. Therefore its Maclaurin series

(2) \[ \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 \]

converges for all \( x \) to the function that is identically zero. However Maclaurin series would not represent the function \( f \). Now we consider the case of complex variable \( z = x + iy \) in (1). That is, let

(3) \[ f(z) = \begin{cases} e^{-1/z^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \]

Then we see that \( f(z) \) could not be continuous at \( z = 0 \). In fact,

(4) \[ f(z) = e^{\frac{-1}{(x+iy)^2}} = e^{\frac{-(x-y)^2}{(x^2+y^2)^2}} = e^{\frac{-(x^2+y^2) + 2xyi}{(x^2+y^2)^2}}, \]

we have

\[ |f(z)| = 1 \text{ for } x = y, \text{ on the other hand } |f(z)| = e^{\frac{1}{x^2}} \text{ for } x = 0 \]

and hence \( |f(z)| \rightarrow \infty \text{ for } y \rightarrow 0 \), in other words \( f(z) \) is discontinuous around the origin in the complex plane \( z = (x, y) \).