\[ \ddot{x} \pm ax \pm bx^3 = 0 \] has a periodic solution associated to the elliptic function

by Akio Arimoto

Department of Mathematics, Tokyo City University

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1. Introduction

We will consider differential equations such as
\[ \ddot{x} + ax + bx^3 = 0, \quad \ddot{x} - ax + bx^3 = 0, \quad \ddot{x} + ax - bx^3 = 0, \quad \ddot{x} - ax - bx^3 = 0, \]
where \( a > 0, b > 0 \), \( x = x(t) \) differentiable real valued function, \( \dot{x} = \frac{dx}{dt} \) and \( \ddot{x} = \frac{d^2x}{dt^2} \). We will show that the solutions of these equations are linear fractional transformations of Jacobi’s elliptic functions \( sn(t,k), cn(t,k) \) and \( dn(t,k) \). If we multiply \( \ddot{x} + ax + bx^3 = 0 \) by \( \dot{x} \) and integrate it in \( t \), we have \( 2\ddot{x}^2 = -2ax^2 - bx^4 + E \) for some integral constant \( E \). We will see show that the solutions of our differential equations have varieties of forms which are dependent on \( E \).

2. Preliminaries

2.1 Cross product

We use a cross product of complex numbers \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) defined by
\[
[\alpha_0, \alpha_1, \alpha_2, \alpha_3] = \frac{\alpha_1 - \alpha_0}{\alpha_1 - \alpha_2} \frac{\alpha_2 - \alpha_3}{\alpha_2 - \alpha_0},
\]
and make linear fractional transformations from this, for example \( F(x) = [\alpha_0, x, \alpha_2, \alpha_3] \frac{x - \alpha_0}{x - \alpha_2} \frac{\alpha_3 - \alpha_2}{\alpha_3 - \alpha_0} \). Now we consider the linear fractional transformation \( \zeta = \frac{Ax + B}{Cx + D} = \varphi(x), AD - BC \neq 0 \). The cross ratio is known to be invariant under linear transformations, that is, for a cross product \([\beta_0, \beta_1, \beta_2, \beta_3] \) of \( \beta_i = \varphi(\alpha_i), i = 0, 1, 2, 3 \), we have \([\alpha_0, \alpha_1, \alpha_2, \alpha_3] = [\beta_0, \beta_1, \beta_2, \beta_3] \).

Also it hold \([x, \alpha_1, \alpha_2, \alpha_3] = [\zeta, \beta_1, \beta_2, \beta_3] \) \([\alpha_0, x, \alpha_2, \alpha_3] = [\beta_0, \zeta, \beta_2, \beta_3] \),

\([\alpha_0, \alpha_1, x, \alpha_3] = [\beta_0, \beta_1, \zeta, \beta_3] \) and \([\alpha_0, \alpha_1, \alpha_2, x] = [\beta_0, \beta_1, \beta_2, \zeta] \).

This last fact can be justified by noticing that a linear fractional
transformation $G(\zeta) = [\beta_0, \zeta, \beta_2, \beta_3]$ satisfy the equality $G \circ \varphi = F$ because $G \circ \varphi(x)$ and $F(x)$ are linear fractional transformations, and both transform $\alpha_0, \alpha_2, \alpha_3$ into $0, \infty, 1$ and a linear transformation is determined uniquely by values at three distinctive points. By the same reason we have

\begin{align*}
(1) & \quad [x, \alpha_1, \alpha_2, \alpha_3] = [\zeta, \beta_1, \beta_2, \beta_3] = \frac{\zeta - \beta_1 \beta_2 - \beta_3}{\zeta - \beta_2 \beta_3 - \beta_1}, \\
(2) & \quad [\alpha_0, x, \alpha_2, \alpha_3] = [\beta_0, \zeta, \beta_2, \beta_3] = \frac{\zeta - \beta_0 \beta_3 - \beta_2}{\zeta - \beta_2 \beta_3 - \beta_0}, \\
(3) & \quad [\alpha_0, \alpha_1, x, \alpha_3] = [\beta_0, \beta_1, \zeta, \beta_3], \\
(4) & \quad [\alpha_0, \alpha_1, \alpha_2, x] = [\beta_0, \beta_1, \beta_2, \zeta].
\end{align*}

However two of (1) – (4) are redundant equations because (1) is equivalent to (3) and (2) is equivalent to (4).

2.2 Differential equations of which Jacobi’s elliptic functions are solutions

Let $0 < k < 1$ and denote for simplicity $s = sn(t, k), c = cn(t, k), d = dn(t, k)$, which are referred to Jacobi's elliptic functions. Then it is well known that $s, c, d$ are combined with the relations $c = \sqrt{1-s^2}, d = \sqrt{1-k^2s^2}, \dot{s} = cd, \dot{c} = -sd, \dot{d} = -k^2sc$. In other words they satisfy the differential equations

\[ \dot{s}^2 = c^2d^2 = (1-s^2)(1-k^2s^2), \]
\[ \dot{c}^2 = s^2d^2 = (1-c^2)((1-k^2)+k^2c^2) \quad \text{and} \quad \dot{d}^2 = k^2s^2c^2 = (1-d^2)(d^2 - (1-k^2)). \]

In addition to these if we set $u = \frac{s}{c}$, we have one more differential equation

\[ \dot{u}^2 = (1+u^2)((1-k^2)+u^2). \]

Now we have the classification of the types (5)-(8) of differential equations associated with $s, c, d, u$:

\begin{align*}
(5) & \quad \dot{s}^2 = (1-s^2)(1-k^2s^2) = k^2(\zeta - \beta_0)(\zeta - \beta_1)(\zeta - \beta_2)(\zeta - \beta_3) = 0, \\
\beta_0 & = 1, \beta_1 = \frac{1}{k}, \beta_2 = -1, \beta_3 = -\frac{1}{k},
\end{align*}
\[
\begin{align*}
[\beta_0, \beta_1, \beta_2, \beta_3] &= \left(\frac{1-k}{1+k}\right)^2 \\
(\beta_1 - \beta_3)^2 (\beta_2 - \beta_0)^2 &= \frac{16}{k^2} \\
[\zeta, \beta_1, \beta_2, \beta_3] &= -\frac{\zeta - \frac{1}{k}}{1-k} \\
&\quad + \frac{1}{1+k} \\
[\beta_0, \zeta, \beta_2, \beta_3] &= \frac{\zeta - 1 - \frac{1}{k}}{\zeta + 1 + \frac{1}{k}} \\
(6\cdot c) \ z^2 &= (1 - \zeta^2)(1 - \frac{1}{k^2}) + k^2 \zeta^2 = -k^2(\zeta - \beta_0)(\zeta - \beta_1)(\zeta - \beta_2)(\zeta - \beta_3), \\
\beta_0 &= 1, \beta_1 = i\frac{\sqrt{1-k^2}}{k}, \beta_2 = -1, \beta_3 = -i\frac{\sqrt{1-k^2}}{k}, \\
[\beta_0, \beta_1, \beta_2, \beta_3] &= \left(\frac{k - i\sqrt{1-k^2}}{k + i\sqrt{1-k^2}}\right)^2 \\
(\beta_1 - \beta_3)^2 (\beta_2 - \beta_0)^2 &= -16 \left(\frac{1-k}{k^2}\right)^2 \\
[\zeta, \beta_1, \beta_2, \beta_3] &= \frac{\zeta - \beta_1 \beta_2 \beta_3}{\zeta - \beta_3 \beta_2 \beta_1} = \frac{\zeta - i\sqrt{1-k^2}}{k + i\sqrt{1-k^2}} \frac{k - i\sqrt{1-k^2}}{k + i\sqrt{1-k^2}} \\
[\beta_0, \zeta, \beta_2, \beta_3] &= \frac{\zeta - \beta_0 \beta_2 \beta_3}{\zeta - \beta_2 \beta_3 \beta_0} = \frac{\zeta - 1 - k i \sqrt{1-k^2}}{\zeta + 1 + k i \sqrt{1-k^2}} \\
(7\cdot d) \ z^2 &= (1 - \zeta^2)(\zeta^2 - (1 - \frac{1}{k^2})) = -(\zeta - \beta_0)(\zeta - \beta_1)(\zeta - \beta_2)(\zeta - \beta_3), \\
\beta_0 &= 1, \beta_1 = \sqrt{1-k^2}, \beta_2 = -1, \beta_3 = -\sqrt{1-k^2}, [\beta_0, \beta_1, \beta_2, \beta_3] = \left(\frac{1-\sqrt{1-k^2}}{1+\sqrt{1-k^2}}\right)^2 \\
(\beta_1 - \beta_3)^2 (\beta_2 - \beta_0)^2 &= 16(1-k^2) 
\end{align*}
\]
\[
[\zeta, \beta_1, \beta_2, \beta_3] = \frac{\zeta - \beta_1 \beta_2 - \beta_3}{\zeta - \beta_3 \beta_2 - \beta_1} = \frac{\zeta - \sqrt{1-k^2}}{\zeta + \sqrt{1-k^2}} \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}
\]

\[
[\beta_0, \zeta, \beta_2, \beta_3] = \frac{\zeta - \beta_0 \beta_2 - \beta_3}{\zeta - \beta_2 \beta_3 - \beta_0} = \frac{\zeta - 1 - \sqrt{1-k^2}}{\zeta + 1 + \sqrt{1-k^2}}
\]

(8·u) \[
\dot{\zeta}^2 = (1 + \zeta^2) \left((1 - k^2) + \zeta^2\right) = \left(\zeta - \beta_0\right)\left(\zeta - \beta_1\right)\left(\zeta - \beta_2\right)\left(\zeta - \beta_3\right).
\]

\[
\beta_0 = i, \beta_1 = i \sqrt{1-k^2}, \beta_2 = -i, \beta_3 = -i \sqrt{1-k^2},
\]

\[
[\beta_0, \beta_1, \beta_2, \beta_3] = \left(\frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}\right)^2
\]

\[
(\beta_1 - \beta_2)^2 (\beta_2 - \beta_0)^2 = 16(1-k^2).
\]

\[
[\zeta, \beta_1, \beta_2, \beta_3] = \frac{\zeta - \beta_1 \beta_2 - \beta_3}{\zeta - \beta_3 \beta_2 - \beta_1} = \frac{\zeta - i \sqrt{1-k^2}}{\zeta + i \sqrt{1-k^2}} \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}
\]

\[
[\beta_0, \zeta, \beta_2, \beta_3] = \frac{\zeta - \beta_0 \beta_2 - \beta_3}{\zeta - \beta_2 \beta_3 - \beta_0} = \frac{\zeta - i \sqrt{1-k^2}}{\zeta + i \sqrt{1-k^2}} \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}
\]

And (5) corresponds to \(sn(t,k)\), (6) to \(cn(t,k)\), (7) to \(dn(u,k)\) and (8) \newline
\(\frac{sn(t,k)}{cn(t,k)}\).

We will apply these classification in order to solve our differential equations.

2.3 Lemma

Let \(x = x(t)\) and \(\zeta = \zeta(t)\) satisfy \(\zeta = \frac{Ax + B}{Cx + D} \equiv f(x)\) and \(\beta_i = f(\alpha_i), \ i = 0,1,2,3\).

Then we have the following lemma.

Lemma 1

\[
(9) \quad \frac{(\alpha_i - \alpha_1)^2 (\alpha_2 - \alpha_0)^2 x^4}{(x - \alpha_0)^2 (x - \alpha_1)^2 (x - \alpha_2)^2 (x - \alpha_3)^2} = \frac{(\beta_i - \beta_1)^2 (\beta_2 - \beta_0)^2 \zeta^4}{(\zeta - \beta_0)^2 (\zeta - \beta_1)^2 (\zeta - \beta_2)^2 (\zeta - \beta_3)^2}
\]

Proof: From the equality \( [x, \alpha_1, \alpha_2, \alpha_3] = [\zeta, \beta_1, \beta_2, \beta_3] \), or
\[
\frac{x - \alpha_1}{x - \alpha_1} \frac{\alpha_2 - \alpha_3}{\zeta - \beta_1} = \frac{\zeta - \beta_1}{\beta_2 - \beta_1}, \text{ we have}
\]
\[
\frac{(\alpha_i - \alpha_1) \hat{x}}{(y - \alpha_1) (\alpha_i - \alpha_2)} = \frac{(\beta_i - \beta_1) \hat{\zeta}}{(\zeta - \beta_1) (\beta_i - \beta_2)}
\]

by differentiation in \( t \). In the same way we have
\[
\frac{(\alpha_0 - \alpha_1) \hat{x}}{(y - \alpha_1) (\alpha_0 - \alpha_2)} = \frac{(\beta_0 - \beta_1) \hat{\zeta}}{(\zeta - \beta_1) (\beta_0 - \beta_2)},
\]
\[
\frac{(\alpha_1 - \alpha_0) \hat{x}}{(y - \alpha_1) (\alpha_1 - \alpha_2)} = \frac{(\beta_1 - \beta_0) \hat{\zeta}}{(\zeta - \beta_1) (\beta_1 - \beta_2)}
\]

and
\[
\frac{(\alpha_2 - \alpha_0) \hat{x}}{(y - \alpha_0) (\alpha_2 - \alpha_0)} = \frac{(\beta_2 - \beta_0) \hat{\zeta}}{(\zeta - \beta_0) (\beta_2 - \beta_0)}.
\]

Multiply these on each sides, we have
\[
\frac{(\alpha_1 - \alpha_1)^2 (\alpha_2 - \alpha_0)^2 (\alpha_3 - \alpha_2)^2 (\alpha_1 - \alpha_0)^2 \hat{x}^4}{(y - \alpha_0)^2 (x - \alpha_1)^2 (x - \alpha_2)^2 (x - \alpha_3) (\alpha_2 - \alpha_2) (\alpha_1 - \alpha_0)^2} = \frac{(\beta_0 - \beta_1)^2 (\beta_2 - \beta_0)^2 (\beta_1 - \beta_2)^2 (\beta_0 - \beta_0)^2 \hat{\zeta}^4}{(\zeta - \beta_1)^2 (\zeta - \beta_1)^2 (\zeta - \beta_0)^2 (\zeta - \beta_2)^2 (\beta_1 - \beta_2)^2 (\zeta - \beta_0)^2}
\]

Canceling out each sides of (10) by \([\alpha_0, \alpha_1, \alpha_2, \alpha_3] = [\beta_0, \beta_1, \beta_2, \beta_3]\), we have the result (9). //

Lemma 2. If \( 2\hat{x}^2 = \pm b (x - \alpha_0)(x - \alpha_1)(x - \alpha_2) (x - \alpha_3), \) then we have
\[
\frac{\hat{\zeta}^4}{(\zeta - \beta_0)^2 (\zeta - \beta_1)^2 (\zeta - \beta_2)^2 (\zeta - \beta_3)^2} = \frac{(\alpha_1 - \alpha_1)^2 (\alpha_2 - \alpha_0)^2}{b^2 (\beta_0 - \beta_1)^2 (\beta_2 - \beta_0)^2}
\]

3. \( \ddot{x} + ax + bx^3 = 0, a > 0, b > 0 \)

We have now
\[
2\hat{x}^2 = -2ax^2 - bx^4 + E,
\]
where E must be nonnegative. Let write the right hand of (11) as
\[
-2ax^2 - bx^4 + E = -b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3),
\]
The cross ratio of \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) is
\[
(14) \quad \left[ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \right] = \left( \frac{\sqrt{c-a} - i \sqrt{c+a}}{\sqrt{c-a} + i \sqrt{c+a}} \right)^2.
\]

We are looking for a linear fractional transformation (cross ratio relation) which associated to the type of differential equations in section 2.2. We select (2) \( \beta_0 = 1, \beta_1 = \sqrt{1-k^2} \), \( \beta_2 = -1, \beta_3 = -i \sqrt{1-k^2} \) among equations (5) to (8) because roots of the right hand side consists of two real numbers and two imaginary numbers. Hence we have
\[
(15) \quad \left[ \beta_0, \beta_1, \beta_2, \beta_3 \right] = \left( \frac{k - i \sqrt{1-k^2}}{k + i \sqrt{1-k^2}} \right)^2 = \left( \frac{\sqrt{c-a} - i \sqrt{c+a}}{\sqrt{c-a} + i \sqrt{c+a}} \right)^2.
\]

If we now take
\[
(16) \quad k = \sqrt{\frac{c-a}{2c}},
\]
then we have
\[
(17) \quad \frac{\sqrt{c-a} - i \sqrt{c+a}}{\sqrt{c-a} + i \sqrt{c+a}} = \frac{k - i \sqrt{1-k^2}}{k + i \sqrt{1-k^2}}.
\]

From (13) and (16) we have \( (\alpha_1 - \alpha_3)^2 (\alpha_2 - \alpha_0)^2 = -16 \frac{c^2-a^2}{b^2} \) and
\[
(\beta_1 - \beta_3)^2 (\beta_2 - \beta_0)^2 = -16 \frac{1-k^2}{k^2} = -16 \frac{c+a}{c-a},
\]
so the relation (9) gets the form
\[
(18) \quad \frac{-16(x^2-a^2)\dot{x}^4}{b^2(x-\alpha_0)^2(x-\alpha_1)^2(x-\alpha_2)^2(x-\alpha_3)^2} = \frac{-16 \frac{c+a}{c-a} k^4 \dot{\zeta}^4}{(1-\zeta^2)^2 \left( (1-k^2) + k^2 \zeta^2 \right)^2}
\]
from which we have
\[
(19) \quad \frac{\dot{\zeta}^4}{(1-\zeta^2)^2 \left( k^2 \zeta^2 + (1-k^2) \right)^2} = c^2,
\]
\[
(20) \quad \frac{d\zeta}{\sqrt{(1-\zeta^2) \left( k^2 \zeta^2 + (1-k^2) \right)}} = \sqrt{c}.
\]
The solution of the differential equation (20) is for any constant $t_0$

\[(17) \quad \zeta(t) = \pm cn\left(\sqrt{c} (t-t_0),k \right).\]

We have to determine a linear fractional transformation of $\zeta$ which is a solution $x$ of the differential equation (11).

From (18), (20) we have

\[(22) \quad \begin{pmatrix} \frac{x-i}{\sqrt{\frac{c+a}{b}}} \\ \frac{x+i}{\sqrt{\frac{c+a}{b}}} \end{pmatrix} \begin{pmatrix} -\frac{c-a}{b} & +i \frac{c+a}{b} \\ -\frac{c-a}{b} & -i \frac{c+a}{b} \end{pmatrix} = \begin{pmatrix} \frac{\zeta-i}{\sqrt{\frac{1-k^2}{k}}} \\ \frac{\zeta+i}{\sqrt{\frac{1-k^2}{k}}} \end{pmatrix} \begin{pmatrix} -1+i \sqrt{\frac{1-k^2}{k}} \\ -1-i \sqrt{\frac{1-k^2}{k}} \end{pmatrix}.\]

From these relations, using (13) we get just one result

\[(24) \quad x(t) = \sqrt{\frac{c-a}{b}} \zeta(t).\]

**Theorem 1.** $2 \ddot{x}^2 = -2ax^2 - bx^4 + E$, $a > 0, b > 0$ has a solution

\[x(t) = \pm \sqrt{\frac{a^2 + Eb - a}{b}} cn\left(\left(a^2 + Eb\right)^{\frac{1}{2}} (t-t_0),k \right),\]

where $t_0$ is arbitrary real number and $k^2 = \frac{\sqrt{a^2 + Eb - a}}{2\sqrt{a^2 + Eb}}$.

4. $\ddot{x} - ax + bx^3 = 0, a > 0, b > 0$

In this case we have

\[(25) \quad 2 \ddot{x}^2 = 2ax^2 - bx^4 + E\]

Since we assume $x$ is real valued, it holds $E \geq -\frac{a^2}{b}$. We will discuss two case,

$E > 0$ and $0 > E \geq -\frac{a^2}{b}$ to find solution of our differential equation. For the case, $E > 0$, we write the right hand side of (25) in the way:

\[2ax^2 - bx^4 + E = -b(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3), \quad c = \sqrt{a^2 + Eb}, \quad \alpha_0 = \sqrt{\frac{c+a}{b}}, \quad \alpha_2 = -\sqrt{\frac{c+a}{b}}, \quad \alpha_1 = i\sqrt{\frac{c-a}{b}}, \quad \alpha_3 = -i\sqrt{\frac{c-a}{b}}, \quad \text{which is the same as (13) except for the} \]
sign of $a$. Hence we have the following theorem,

**Theorem 2.** $2\dot{x}^2 = 2ax^2 - bx^4 + E \quad a > 0, b > 0, E > 0$ has a solution

$$x(t) = \pm \sqrt{\frac{a^2 + Eb + a}{b}} \operatorname{cn}\left(\frac{1}{2}(a^2 + Eb)^{1/2}(t-t_0), k\right),$$

where $t_0$ is arbitrary real number and $k^2 = \frac{a^2 + Eb + a}{2\sqrt{a^2 + Eb}}$

If we let $E \to 0$ in Theorem 2, then we have the following corollary

**Corollary 3.** $2\dot{x}^2 = 2ax^2 - bx^4, a > 0, b > 0$ has a solution

$$x(t) = \pm \sqrt{\frac{2a}{b}} \operatorname{sech}\left(\sqrt{a}(t-t_0)\right)$$

Now we assume that $0 > E \geq -\frac{a^2}{b}$.

(26) $2\dot{x}^2 = 2ax^2 - bx^4 + E = -b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$

where

(27) $c = \sqrt{a^2 + Eb}, \alpha_0 = \sqrt{\frac{a + c}{b}}, \alpha_1 = -\sqrt{\frac{a + c}{b}}, \alpha_2 = \sqrt{\frac{a - c}{b}}, \alpha_3 = -\sqrt{\frac{a - c}{b}}.$

From (26) it must be $\sqrt{\frac{a - c}{b}} < x < \sqrt{\frac{a + c}{b}}, -\sqrt{\frac{a + c}{b}} < x < -\sqrt{\frac{a - c}{b}}$, so we apply (7) to look for solution and letting

(28) $\beta_0 = 1, \beta_2 = -1, \beta_1 = \sqrt{1-k^2}, \beta_3 = -\sqrt{1-k^2},$

then we have $[\alpha_0, \alpha_1, \alpha_2, \alpha_3] = \left(\frac{\sqrt{a + c} - \sqrt{a - c}}{\sqrt{a + c} + \sqrt{a - c}}\right)^2$ and $[\beta_0, \beta_1, \beta_2, \beta_3] = \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}} \left(\frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}\right)^2$, which are equal, so we have

(28) $\frac{\sqrt{a + c} - \sqrt{a - c}}{\sqrt{a + c} + \sqrt{a - c}} = \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}}.$

From which we have
(29) \(1 - k^2 = \frac{a - c}{a + c}, \quad k^2 = \frac{2c}{a + c}\).

Using these, the equality (9) for \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) in (27) get into

\[
\frac{16(a^2 - c^2)x^4}{b^2(x - \alpha_0)^2(x - \alpha_1)^2(x - \alpha_2)^2(x - \alpha_3)^2} = \frac{16\sqrt{1-k^2}\zeta^4}{(1-\zeta^2)^2(\zeta^2-(1-k^2))}
\]

or

\[
\frac{\zeta^4}{(1-\zeta^2)^2(\zeta^2-(1-k^2))} = \frac{(a + c)^2}{4}.
\]

Thus we have

(30) \(\zeta(t) = \pm dn\left(\sqrt{\frac{a + c}{2}t, k}\right)\).

Both relations (1) and (3) gives us the relation

(31) \(x = \sqrt{\frac{a + c}{b}}\zeta\).

In fact the equation (1) is

\[
\begin{pmatrix}
\frac{x - \frac{a - c}{b}}{x + \frac{a - c}{b}} \\
\frac{-\frac{a + c}{b} + \sqrt{\frac{a - c}{b}}}{-\frac{a + c}{b} - \sqrt{\frac{a - c}{b}}}
\end{pmatrix}
\begin{pmatrix}
\zeta - \sqrt{1-k^2} \\
\zeta + \sqrt{1-k^2}
\end{pmatrix}
\begin{pmatrix}
-1 + \sqrt{1-k^2} \\
-1 - \sqrt{1-k^2}
\end{pmatrix}
\]

from which we have

(32) \(\frac{x - \sqrt{\frac{a - c}{b}}}{x + \sqrt{\frac{a - c}{b}}} = \frac{\zeta - \sqrt{1-k^2}}{\zeta + \sqrt{1-k^2}}\).

It is easily seen (32) is equivalent to (31). Hence we have the following theorem.

**Theorem 4.** \(2x^2 = 2ax^2 - bx^4 + E\), \(0 > E \geq -\frac{a^2}{b}\), \(a > 0, b > 0\), has a solution, for any real \(t_0\)

\[
x(t) = \pm\sqrt{\frac{a^2 + Eb + a}{b}} dn\left(\sqrt{\frac{a^2 + Eb + a}{2}}(t - t_0), k\right)
\]

where \(k^2 = \frac{2\sqrt{a^2 + Eb}}{\sqrt{a^2 + Eb + a}}\).
We also get Corollary 3 from Theorem 4 when \( E \to 0 \).

5. \( \ddot{x} + ax - bx^3 = 0, a > 0, b > 0 \)

We devide the problem type into the three cases: \( 0 < E < \frac{a^2}{b}, \frac{a^2}{b} < E \)

If we have \( E < 0 \), we set

\[
2\ddot{x} = -2ax^2 + bx^4 + E = b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3),
\]

where

\[
\alpha_0 = \sqrt{a^2 - Eb}, \alpha_1 = \frac{c + a}{b}, \alpha_2 = -\frac{c - a}{b}, \alpha_3 = -i\sqrt{\frac{c - a}{b}}.
\]

Comparing (12)(13) and (33)(34), we notice that we get the same equation if we change \( a \) by \(-a\) and \( E \) by \(-E\). However there is a difference \( x \) in (33) must satisfy

\[
x^2 > \left( \frac{c + a}{b} \right)^2
\]

whereas \( x \) in (12) must satisfy \( x^2 < \left( \frac{c - a}{b} \right)^2 \). So we should take the differential equation instead of (20):

\[
\frac{d\zeta}{\sqrt{1 - \zeta^2 (1 - k^2)}} = i\sqrt{c}
\]

and its solution:

\[
\zeta(t) = \pm \text{cn} \left( i\sqrt{c} (t - t_0), k \right) = \frac{\pm 1}{\text{cn} \left( \frac{1}{\sqrt{c} (t - t_0)}, k' \right)}, \quad k'^2 = 1 - k^2
\]

**Theorem 5** \( 2\ddot{x} = -2ax^2 + bx^4 + E \) \((a > 0, b > 0), E < 0 \) has a solution, for any real \( t_0 \)

\[
x(t) = \pm \sqrt{\frac{a^2 - Eb + a}{b}} \frac{1}{\text{cn} \left( \frac{a^2 - Eb}{b} (t - t_0), k \right)} + \frac{\sqrt{a^2 - Eb - a}}{2\sqrt{a^2 - Eb}}
\]

Next if we have \( 0 < E < \frac{a^2}{b} \), we factorize the right side in the following way:

\[
2\ddot{x} = b \left( x^2 - \frac{a}{b} \right)^2 - \left( \frac{a^2 - Eb}{b} \right) = b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3),
\]

where
\[ c = \sqrt{a^2 - Eb}, \alpha_0 = \frac{a-c}{b}, \alpha_1 = \frac{a+c}{b}, \alpha_2 = -\frac{a-c}{b}, \alpha_3 = -\frac{a+c}{b}. \]

We select (5) as the type of differential equation, or we let

\[ \beta_0 = 1, \beta_1 = \frac{1}{k}, \beta_2 = -1, \beta_3 = -\frac{1}{k}. \]

Then cross ratios of (38) and (39) are respectively

\[ [\alpha_0, \alpha_1, \alpha_2, \alpha_3] = \left(\frac{\sqrt{a+c} - \sqrt{a-c}}{\sqrt{a+c} + \sqrt{a-c}}\right)^2, \quad [\beta_0, \beta_1, \beta_2, \beta_3] = \left(\frac{1-k}{1+k}\right)^2. \]

Hence we could have

\[ k = \frac{a-c}{a+c}, \quad \frac{\sqrt{a+c} - \sqrt{a-c}}{\sqrt{a+c} + \sqrt{a-c}} = \frac{1-k}{1+k}. \]

With respect to the relation between \( x \) and \( \zeta \), we have two possibilities. From (1) and (3) we have

\[ x = \frac{a-c}{b}, \quad x = \frac{a-c}{b} \zeta, \]

respectively. From equations

\[ (\alpha_1 - \alpha_3)^2 (\alpha_2 - \alpha_0)^2 = 16 \frac{a^2 - c^2}{b^2}, \quad (\beta_1 - \beta_3)^2 (\beta_2 - \beta_0)^2 = \frac{16}{k^2} \]

we have

\[ \frac{16 \left( a^2 - c^2 \right) x^4}{b^2 (x - \alpha_0)^2 (x - \alpha_1)^2 (x - \alpha_2)^2 (x - \alpha_3)^2} = \frac{16k^2 \zeta^4}{(1 - \zeta^2)^2 (1 - k^2)^2 + k^2 \zeta^2}, \]

or

\[ \frac{d\zeta}{\sqrt{(1 - \zeta^2)(1 - k^2)^2 + k^2 \zeta^2}} = \sqrt{\frac{a+c}{2}} dt, \]

from which we have the solution of (43)

\[ \zeta(t) = \pm sn\left(\sqrt{\frac{a+c}{2}} (t - t_0), k\right). \]

**Theorem 6.** \( 2x^2 = -2ax^2 + bx^4 + E \quad (a > 0, b > 0), \quad 0 < E < \frac{a^2}{b} \) has a solution, for any
real $t_0$, 

$$x(t) = \pm \sqrt{\frac{a - \sqrt{a^2 - Eb}}{b}} \text{sn} \left( \sqrt{\frac{a + \sqrt{a^2 - Eb}}{2}} (t - t_0), k \right)$$

and

$$x(t) = \pm \sqrt{\frac{a - \sqrt{a^2 - Eb}}{b}} \left( \frac{1}{\text{sn} \left( \sqrt{\frac{a + \sqrt{a^2 - Eb}}{2}} (t - t_0), k \right)} \right),$$

where $k^2 = \frac{a - \sqrt{a^2 - Eb}}{a + \sqrt{a^2 - Eb}}$.

Finally if we have $\frac{a^2}{b} < E$, we factorize the right side in the following way:

(44) $2x^2 = b \left( x^2 - \frac{a}{b} \right) + \left( \frac{Eb - a^2}{b} \right) = b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$,

where $\alpha_0 = \overline{\gamma}, \alpha_1 = -\gamma, \alpha_2 = \gamma, \alpha_3 = -\overline{\gamma}, \gamma = \frac{\sqrt{Eb - a + i\sqrt{Eb + a}}}{\sqrt{2b}}$. We can define $\theta$ as $\tan \theta = \frac{\sqrt{Eb - a}}{\sqrt{Eb + a}}$, or $\gamma = |\gamma| e^{i\theta}$. We select (8) as the type of differential equation, or we let

(45) $\beta_0 = i, \beta_1 = i\sqrt{1 - k^2}, \beta_2 = -i, \beta_3 = -i\sqrt{1 - k^2}, \quad [\beta_0, \beta_1, \beta_2, \beta_3] = \left\{ \frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}} \right\}^2$.

Since $[\alpha_0, \alpha_1, \alpha_2, \alpha_3] = [\beta_0, \beta_1, \beta_2, \beta_3]$, we have $\frac{(\gamma + \overline{\gamma})^2}{4|\gamma|^2} = \left( \frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}} \right)^2 = \cos^2 \theta$

from which we have

(46) $1 - k^2 = \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right)^2 = \tan^2 \frac{\theta}{2}$. 

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Then we have
\[(\alpha_1 - \alpha_3)^2 (\alpha_2 - \alpha_0)^2 = |\gamma - \gamma'|^4 = 16|\gamma|^4 \cos \theta \quad \text{and} \quad (\beta_1 - \beta_3)^2 (\beta_2 - \beta_0)^2 = 16(1 - k^2).\]

By Lemma 1, we have
\[
\frac{\zeta^4}{(1 + \zeta^2)((1 - k^2) + \zeta^2)} = \frac{|\gamma|^4 \cos^4 \theta}{(1-k^2)} \quad \frac{x^4}{(x-\alpha_0)^2 (x-\alpha_1)^2 (x-\alpha_2)^2 (x-\alpha_3)^2} = \frac{b^2 |\gamma|^4 \cos^4 \theta}{4(1-k^2)}.
\]

Hence if we put \[\Theta^t = \frac{b^2 |\gamma|^4 \cos^4 \theta}{4(1-k^2)},\]
we have
\[
(46) \quad \zeta(t) = \pm \frac{\sin(\Theta(t-t_0), k)}{\cos(\Theta(t-t_0), k)}.
\]

We determine the relation between \(x\) and \(\zeta\). If we use the equation (1), we have
\[
(47) \quad \frac{x + \gamma - \gamma'}{2|\gamma|} = \zeta - i\sqrt{1-k^2} = \frac{\zeta + i\sqrt{1-k^2}}{1+\sqrt{1-k^2}},
\]
or we have
\[
(48) \quad \frac{x + \gamma - \gamma'}{2|\gamma|} e^{-i\theta} = \zeta - i\sqrt{1-k^2}.
\]

Using the relation (46), we have
\[
(49) \quad x = \frac{|\gamma| \tan \frac{\theta}{2} \zeta + \sqrt{1-k^2}}{\tan \frac{\theta}{2} \zeta - \sqrt{1-k^2}} = \frac{|\gamma| \zeta + \tan \frac{\theta}{2}}{\zeta - \tan \frac{\theta}{2}}.
\]

Next we use the relation (2), we have
\[
(50) \quad \frac{x - \gamma - \gamma'}{2\gamma'} = -\frac{\zeta - i}{\zeta + i},
\]
or we have
\[
(51) \quad \frac{x - \gamma - \gamma'}{2\gamma'} e^{2i\theta} = -\frac{\zeta - i}{\zeta + i}.
\]

From which we have the same formula as (49). Thus we obtain following theorem.

**Theorem 7.** \[2x^2 = -2ax^2 + bx^4 + E \quad (a > 0, b > 0), \quad \frac{a^2}{b} < E\] has a solution, for any real
\[ t_0, x(t) = \frac{\left( \frac{E}{b} \right)^{1/3} \text{sn}(\Theta(t-t_0),k) + \tan \frac{\theta}{2} \text{cn}(\Theta(t-t_0),k)}{\text{sn}(\Theta(t-t_0),k) - \tan \frac{\theta}{2} \text{cn}(\Theta(t-t_0),k)}, \]

\[ \Theta^2 = \frac{b^2}{4} \left[ \cos^2 \theta \right] \quad \text{and} \quad k^2 = \frac{4 \cos \theta}{(1 + \cos \theta)^2}. \]

5. \( \ddot{x} - ax - bx^3 = 0, a > 0, b > 0 \)

For \( 2x^2 = 2ax^2 + bx^4 + E \), we treat three cases distinctively as in section 4:

- \( E < 0, 0 < E < \frac{a^2}{b}, \frac{a^2}{b} < E \).

a) \( E < 0 \)

\[ 2x^2 = 2ax^2 + bx^4 + E = b \left( x^2 + \frac{\sqrt{a^2 - Eb + a}}{b} \right) \left( x + \frac{\sqrt{a^2 - Eb - a}}{b} \right) \left( x - \frac{\sqrt{a^2 - Eb - a}}{b} \right) \]

In this case we can use the result of Theorem 1. We only need to change \( t \) by \( it \) in the formula \( \ddot{x} - ax - bx^3 = 0, a > 0, b > 0 \). Then we have \( 2x^2 = -2ax^2 - bx^4 - E \). Thus if we change \( t \) by \( it \) and \( E \) by \( -E \) in Theorem 1 and use the equation \( \text{cn}(it,k) = \frac{1}{\text{cn}(t,\sqrt{1-k^2})} \), we have

**Theorem 8.** \( 2x^2 = -2ax^2 - bx^4 + E \), \( a > 0, b > 0 \) has a solution

\[ x(t) = \pm \sqrt{\frac{a^2 - Eb - a}{b}} \frac{1}{\text{cn} \left( \frac{1}{\sqrt{(a^2 - Eb)^2} (t-t_0),k} \right)}, \]

where \( t_0 \) is arbitrary real number and \( k^2 = \frac{\sqrt{a^2 - Eb + a}}{2 \sqrt{a^2 - Eb}} \).

b) \( 0 < E < \frac{a^2}{b} \)
\[2x^2 = 2ax^2 + bx^4 + E = b \left( x^2 + \frac{a + \sqrt{a^2 - Eb}}{b} \right) \left( x^2 + \frac{a - \sqrt{a^2 - Eb}}{b} \right) \]

Let \( c = \sqrt{a^2 - Eb} \). Then we have

\[2x^2 = 2ax^2 + bx^4 + E = b \left( x^2 + \frac{a + c}{b} \right) \left( x^2 + \frac{a - c}{b} \right) \]

If we put \( \alpha_0 = i\sqrt{\frac{a-c}{b}} \), \( \alpha_1 = i\sqrt{\frac{a+c}{b}} \), \( \alpha_2 = -i\sqrt{\frac{a-c}{b}} \), \( \alpha_3 = -i\sqrt{\frac{a+c}{b}} \), we have

\[2x^2 = b \left( x^2 + \frac{a+c}{b} \right) \left( x^2 + \frac{c-a}{b} \right) = b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3), \]

\[\left[ \alpha_0, \alpha_1, \alpha_2, \alpha_3 \right] = \left( \frac{\sqrt{a+c} - \sqrt{a-c}}{\sqrt{a+c} + \sqrt{a-c}} \right)^2 \]

and

\[(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_0)^2 = 16\frac{a^2 - c^2}{b^2}.\]

Setting \( \beta_0 = i \), \( \beta_1 = i\sqrt{1-k^2} \), \( \beta_2 = -i \), \( \beta_3 = -i\sqrt{1-k^2} \), we have

\[\left( 1 - \sqrt{1-k^2} \right)^2 = \left( \frac{\sqrt{a+c} - \sqrt{a-c}}{\sqrt{a+c} + \sqrt{a-c}} \right)^2.\]

Thus we can put \( k^2 = \frac{2c}{a+c} \), \( 1-k^2 = \frac{a-c}{a+c} \).

From Lemma 1,

\[\frac{16\frac{a^2 - c^2}{b^2}x^4}{(x - \alpha_0)^2(x - \alpha_1)^2(x - \alpha_2)^2(x - \alpha_3)^2} = \frac{16(1-k^2)x^4}{(1+\zeta^2)^2 \left( (1-k^2) + \zeta^2 \right)^2},\]

or we have

\[\frac{\zeta^4}{(1+\zeta^2)^2 \left( (1-k^2) + \zeta^2 \right)^2} = \left( \frac{a+c}{4} \right)^2.\]

Hence it holds
\[\zeta(t) = \pm \frac{\sqrt{\frac{a+c}{2}(t-t_0),k}}{cn\left(\frac{a+c}{2}(t-t_0),k\right)} \cdot \zeta(t) = \pm i \frac{\sqrt{\frac{a+c}{2}(t-t_0),k}}{cn\left(\frac{a+c}{2}(t-t_0),k\right)}\]

Since
\[\left[x, \alpha_1, \alpha_2, \alpha_3\right] = -\frac{x-i \sqrt{\frac{a+c}{b}}}{x+i \sqrt{\frac{a+c}{b}}} \sqrt{a+c-\sqrt{a-c}} \sqrt{a+c+\sqrt{a-c}}\]

and
\[\left[\zeta, \beta_1, \beta_2, \beta_3\right] = \frac{\zeta-i \sqrt{1-k^2}}{\zeta+i \sqrt{1-k^2}} \frac{1-1-1-k^2}{1+1-k^2}, \text{ we have } -\frac{x-i \sqrt{\frac{a+c}{b}}}{x+i \sqrt{\frac{a+c}{b}}} = \frac{\zeta-i \sqrt{1-k^2}}{\zeta+i \sqrt{1-k^2}},\]

from which we have
\[x(t) = -\sqrt{\frac{a+c}{b}} \sqrt{1-k^2} \frac{1}{\zeta(t)} = -\sqrt{\frac{a-c}{b}} \frac{1}{\zeta(t)}\]

Next since \[\left[\alpha_0, x, \alpha_2, \alpha_3\right] = -\frac{x-i \sqrt{\frac{a-c}{b}}}{x+i \sqrt{\frac{a-c}{b}}} \sqrt{a+c-\sqrt{a-c}} \sqrt{a+c+\sqrt{a-c}}\], we

Have \[\frac{x-i \sqrt{\frac{a-c}{b}}}{x+i \sqrt{\frac{a-c}{b}}} = \frac{\zeta-1}{\zeta+1}, \text{ from which we have also } x(t) = i \sqrt{\frac{a-c}{b}} \zeta(t)\]

**Theorem 9.** \[2x^2 = 2ax^2 + bx^4 + E (a > 0, b > 0), \quad 0 < E < \frac{a^2}{b}\] has a solution, for any real 
\[t_0, x(t) = \pm \frac{\sqrt{a-\sqrt{Eb-a^2}}}{b} \frac{sn\left(\frac{a+\sqrt{Eb-a^2}}{2}(t-t_0),k\right)}{cn\left(\frac{a+\sqrt{Eb-a^2}}{2}(t-t_0),k\right)}\]
and \( x(t) = \pm \sqrt{\frac{a - \sqrt{ab - a^2}}{b}} \left( \frac{a + \sqrt{Eb - a^2}}{2} (t - t_0), k \right) \)

where \( k^2 = \frac{2a^2 - Eb}{a + \sqrt{ab - a^2}} \).

c) \( \frac{a^2}{b} < E \)

\[
2x^2 = 2ax^2 + bx^4 + E = b \left( x^2 + \frac{a}{b} + i\sqrt{E - a^2/b} \right) \left( x^2 + \frac{a}{b} - i\sqrt{E - a^2/b} \right)
\]

\[
= b(x - \alpha_0)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)
\]

where \( \alpha_0 = \gamma, \alpha_1 = -\gamma, \alpha_2 = \gamma, \alpha_3 = -\gamma, \gamma = \frac{\sqrt{E + a + i\sqrt{E - a}}}{\sqrt{2b}} \). We can define \( \theta \)

as \( \tan \theta = \frac{\sqrt{E - a}}{\sqrt{E + a}} \), or \( \gamma = |\gamma|e^{i\theta} \). From now on we can have a same discussion as

Theorem 7 only by changing \( a \) into \( -a \). We have thus

**Theorem 10**

\[
2x^2 = 2ax^2 + bx^4 + E \quad (a > 0, b > 0) \quad , \quad \frac{a^2}{b} < E \quad \text{has a solution}, \quad \text{for any real}
\]

\[
t_0, x(t) = \frac{\left( \frac{E}{b} \right)^{\frac{1}{2}} \left( \frac{1}{2} \right) \left( \Theta(t - t_0), k \right) + \tan \frac{\theta}{2} cn(\Theta(t - t_0), k)}{sn(\Theta(t - t_0), k) - \tan \frac{\theta}{2} cn(\Theta(t - t_0), k)},
\]

\[
\Theta^4 = \frac{b^2 |\gamma|^4 \cos^4 \theta}{4(1 - k^2)} \quad \text{and} \quad k^2 = \frac{4 \cos \theta}{(1 + \cos \theta)^2}.
\]